

where  $m_{\min}$  is the minimum unstable wave number as derived from the linear theory, i.e.,  $m_{\min}^2 = 4\nu_{i\text{eff}}\nu_{e\text{eff}}/\omega_r^2$ . The function  $\xi(\rho)$  is sensitive to the behavior of  $g(\theta)$  near  $\theta=0$ . The calculated function<sup>2</sup> falls steeply to zero at  $\theta=0$ ; but, taking into account the effect of untrapped particles, it seems more reasonable to treat  $g(0)$  as finite in evaluating  $\xi$ . For reasonable parameters like  $m_{\min}=4$ ,  $\bar{m}=7$ , we then find  $\rho \equiv e\bar{\varphi}/\epsilon T \approx 5 \times 10^{-2}$ . Our conclusion is that a small level of fluctuations is sufficient to stabilize the collisional instability through nonlinear detrapping of the ions.

Other estimates<sup>3,4</sup> use nonlinear radial diffusion as the leading stabilizing mechanism. We find that our estimates are more optimistic since the detrapping effect saturates the instability at a lower level. By defining a turbulent diffusion coefficient<sup>3,4</sup> as

$$D_T = \epsilon^{1/2} \gamma (E_\theta / \omega B)^2,$$

and taking  $E_\theta \approx \bar{m}\bar{\varphi}/a$ , we find

$$D_T = \epsilon^{1/2} (\bar{m}T/eBa)^2 (e\bar{\varphi}/\epsilon T)^2 \nu_e^{-1},$$

which is usually significantly smaller than the other estimates.<sup>3,4</sup> We finally note that a dynamical stabilization scheme,<sup>5</sup> by increasing  $m_{\min}$  and therefore decreasing  $e\bar{\varphi}/\epsilon T$ , could further reduce this turbulent diffusion.

In conclusion, we note that the preceding study depends on the validity of the random-walk movement for the barely trapped particles. The actual movement is difficult to evaluate since the different time scales  $\omega$  and  $\omega_{bi}$  are comparable. Furthermore, it depends on the value  $g(0)$  of the fluctuations near the mirror point; additional work is in progress along these lines.

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## Transport Properties of a Three-Dimensional Van der Waals Fluid

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We have made a microscopic calculation of the transport coefficients  $X$  of a Van der Waals fluid with an interaction potential  $V(r) = V^S(r) + \gamma^3 V^L(\gamma r)$ . After performing a "hydrodynamical renormalization" of the propagators of the theory, we can show that the first correction to  $X$  due to  $V^L$  is of order  $\gamma$  when  $\gamma$  is small. Explicit, model-independent expressions for this first correction are given.

In spite of the considerable development of non-equilibrium statistical mechanics within the last twenty years, the progress in the classical transport theory of a three-dimensional fluid has remained very slow and a review of the solvable models is readily made: One has the dilute-gas model (the Boltzmann equation and its extension to higher densities, with, however, the well-known logarithmic difficulty appearing already at the second correction), the hot-plasma model (the Balescu-Lenard-Gurnsey kinetic equation),

and the Brownian-motion problem (the Fokker-Planck equation).<sup>1</sup> The solvability of each of these models is of course intimately connected with the existence of a smallness parameter—the density, charge, and inverse mass, respectively, of the heavy particle—and, in each case, a well-known equilibrium analog exists (which is trivial in the classical Brownian-motion problem).

Yet, almost one century ago,<sup>2</sup> Van der Waals invented another simple equilibrium model which turned out to be extremely fruitful in describing

qualitatively the main properties of realistic fluids. In modern language, a rigorous formulation of this model can be given by writing the pair interaction in the form<sup>3</sup>

$$V(r) = V^S(r) + \gamma^3 V^L(\gamma r), \quad (1)$$

where  $V^S$  refers to the short-range reference system while  $V^L$  is the long-range part of the potential, with range  $\gamma^{-1}$ . One then analyzes the properties of the system in the limit  $\gamma \rightarrow 0$ . In particular, this  $\gamma \rightarrow 0$  limit leads to the Van der Waals-type equation combined with the Maxwell equal-area construction.

It is rather surprising that, for transport properties, this Van der Waals model has not yet received much attention, except for a few semiphenomenological studies<sup>4</sup> close to the critical point (where precisely, for  $\gamma$  finite, the  $\gamma$  expansion of equilibrium properties is known to fail!) It is the aim of the present note to report an attempt we have made to analyze correctly transport properties of the one-phase Van der Waals fluid. Our main result is an expression of the first correction to the various transport coefficients, hereafter generically denoted by  $X$  ( $X = \eta, \kappa, \xi, D$ ); let us stress that, in contrast to the equilibrium case, there is no  $\gamma^0$  term due to the long-range potential  $V^L(\gamma r)$ . The details of the calculations, which are fairly involved, will be published elsewhere, and we limit ourselves here to an outline of the method and to a presentation of the final results.

The starting point of our calculation is the Green-Kubo formula for the transport coefficients  $X$ .<sup>5</sup> Following the method presented earlier,<sup>6</sup> we make a formal (infinite-order) perturbative analysis of these quantities utilizing the many-body techniques developed in Ref. 1. This allows us to reduce the calculation of  $X$  to the analysis of the basic operators which always appear in kinetic theory: In particular, the linearized generalized Boltzmann operator, denoted by  $\Psi_0^i$ , plays a fun-

damental role in this formulation; and we shall illustrate the main feature of our method by discussing this operator only. We shall not need its explicit form here; let us simply point out that it is given, through well-defined rules, by an infinite-order perturbation expansion involving the Liouville operator,  $i\{V, \}$ , associated with the potential  $V$  of Eq. (1). Because, the long-range part  $V^L$  is in some sense "weak," it is, of course, tempting to expand  $\Psi_0^i$  in a naive perturbation expansion in powers of  $\gamma$ ; yet, one immediately finds that this straightforward expansion is divergent in the  $\gamma \rightarrow 0$  limit. The physical reason for this difficulty is easy to understand: In a perturbative description, the motion of the particles between two successive interactions is treated as free;  $\Psi_0^i$  is thus a functional of the unperturbed propagator  $G_q^0(v_1)$ ,<sup>1</sup>

$$\Psi_0^i = \Psi_0^i(\{G_q^0\}), \quad (2)$$

$$G_q^0(v_1) = \lim_{\epsilon \rightarrow 0} (i\epsilon - iqv_1)^{-1}. \quad (3)$$

Yet, the values of  $q$  which are excited by the long-range potential  $V^L$  are small [ $q \sim O(\gamma)$ ], which means that Eq. (3) then describes a particle propagating freely over long distances, of the order  $\gamma^{-1}$ . As we are in a dense system, particles are, however, in continuous interaction with each other and, instead of (3), we should consider the "renormalized" propagation expressed by  $G_q(v_1)$ ,

$$G_q(v_1) = \lim_{\epsilon \rightarrow 0} [i\epsilon - iqv_1 + i\Psi_q^i(v_1)]^{-1}, \quad (4)$$

where  $i\Psi_q^i$  describes the collisions of particle 1 with the medium (to arbitrary order in the density). We have now, instead of (2),

$$\Psi_0^i = \tilde{\Psi}_0^i(\{G_q\}). \quad (5)$$

It is a remarkable property of  $G_q$  that, even for  $q \sim O(\gamma)$  (which cannot be considered as the strict hydrodynamic regime because  $q$  is not the smallest parameter of the problem), the dominant contribution to (4) can be shown to be

$$\lim_{\substack{q \rightarrow 0, \gamma \rightarrow 0, \\ y = q\gamma^{-1} \text{ finite}}} G_q(v_1) = \sum_{\alpha=1}^5 |f_\alpha\rangle [i\epsilon + \Lambda_\alpha^q(\gamma)]^{-1} \langle f_\alpha'|, \quad (6)$$

where the  $\Lambda_\alpha^q(\gamma)$  are generalized hydrodynamical eigenvalues to be defined below, while  $|f_\alpha\rangle$  and  $\langle f_\alpha'|$  are the corresponding one-particle eigenfunctions. Taking  $q$  along the  $x$ -axis, we have

$$\Lambda_{1,2}^q(\gamma) = \pm ic(y)q - \Gamma(y)q^2; \quad \Lambda_{3,4}^q(\gamma) = -\eta^S q^2/n; \quad \Lambda_5^q(\gamma) = -\kappa^S q^2/nC_p(y), \quad (7)$$

where the following definitions have been used:

$$C^2(y) = \tilde{\gamma}(y)m^{-1}/\chi_T(y); \quad \tilde{\gamma}(y) = C_p(y)/C_v^S; \quad \chi_T(y) = n^{-1} \left[ \left( \frac{\partial p}{\partial n} \right)_T + nV_y^L \right]^{-1};$$

and

$$C_p(y) = C_p^S + \{T[(\partial p/\partial T)_n^S]^2[\chi_T(y) - \chi_T^S]n^{-1}\}; \quad \Gamma(y) = \{\frac{4}{3}\eta^S + \xi^S + [1/C_p(y) - 1/C_p^S]\kappa^S\}/2n. \quad (8)$$

In these formulas, we have followed the standard notations for thermodynamic and transport coefficients; moreover quantities indexed by the superscript  $S$  refer to the short-range reference system and are obtained by formally setting  $V^L(\gamma r) \equiv 0$  in the interaction potential (1); finally  $V_y^{-L}$  is the Fourier transform of  $V^L(x)$ . One sees readily that the modes (7) can be interpreted as a finite  $y = q\gamma^{-1}$  generalization to a Van der Waals fluid of the usual hydrodynamical modes.<sup>7</sup>

Using (5) and (6), we have been able to show that, to arbitrary order in the density,  $\Psi_0^I$  can be expanded in powers of  $\gamma$ :

$$\Psi_0^I = \Psi_0^{I,S} + \sum_{n=1}^{\infty} \gamma^n \tilde{\Psi}_0^{I(n)}(\{G_q\}). \quad (9)$$

The other operators which enter the many-body analysis of  $X$  can be treated along the same lines and, parallel to (9), we get an expansion for the transport coefficients:

$$X = X^S + \gamma \delta X_1 + \dots \quad (10)$$

Although the explicit evaluation of the higher-order terms seems presently out of the question, it is nevertheless possible to obtain a compact *model-independent* formula for the first-order term  $\delta X_1$ ; we use for this a generalization to arbitrary density of the method followed—but only to lowest order in the density—by Dorfman and Cohen.<sup>8</sup> We give here the final result for the shear viscosity:

$$\gamma \delta \eta_1 = \frac{\gamma k T}{30\pi^2} \int_0^\infty dy \left\{ \left[ \left( 1 + \frac{1}{2c^2(y)} n y \frac{\partial V_y^L}{\partial y} \right)^2 \frac{1}{\Gamma(y)} \right] - \frac{1}{\Gamma^S} + \left( n \chi_T(y) \frac{\tilde{\gamma}(y) - 1}{\tilde{\gamma}(y)} n y \frac{\partial V_y^L}{\partial y} \right)^2 \frac{n C_p(y)}{2\kappa^S} \right\}; \quad (11)$$

and for the thermal conductivity:

$$\gamma \delta \kappa_1 = \frac{\gamma k}{6\pi^2} \int_0^\infty dy \left\{ \left[ \left( 1 + \frac{1}{2c^2(y)} n y \frac{\partial V_y^L}{\partial y} \right)^2 \frac{c^2(y)}{2\Gamma(y)} \right] - \frac{(c^S)^2}{2\Gamma^S} + 2Tn \left( \frac{C_p(y)}{\eta^S + \kappa^S/C_p(y)} - \frac{C_p^S}{\eta^S + \kappa^S/C_p^S} \right) \right\}. \quad (12)$$

We have also obtained the corresponding result for the bulk viscosity, but this formula, involving derivatives of  $C_v$ ,  $C_p$ , and  $\chi_T$ , is too long to be reported here, while the first-order correction identically vanishes for the self-diffusion  $D$ . The remarkable feature of these equations is that, apart from macroscopic properties of the reference system, they only involve the potential  $V_y^L$  inside a simple quadrature.

We conclude with a few remarks of general interest:

(1) Although the  $N$ -body method followed here is by no means rigorous in the mathematical sense of the word, our calculation is formally exact<sup>7</sup> and no approximation has been made in the derivation of Eqs. (9) and (10).

(2) As can be seen from the final results (11) and (12), or perhaps better by the hydrodynamical expansion (6), our calculation is essentially based on a microscopic mode-mode-coupling description. To the best of our knowledge, the present analysis is the first microscopic justification, valid to arbitrary order in the density (see, however, Ref. 8), of this type of method, which is usually based on macroscopic considerations.<sup>9,10</sup>

(3) Because Eqs. (11) and (12) only involve macroscopic quantities (suitably generalized to finite  $y = q\gamma^{-1}$ ), it can be guessed *a priori* that macroscopic arguments (suitably adapted to finite  $y$ ) could lead to the same result. As a matter of fact, the attentive reader will have already noticed the analogy between our Eqs. (11) and (12) and the contribution to the Green-Kubo integrands, which gives rise to the well-known  $t^{-3/2}$  long-time behavior.<sup>11</sup> (In particular, the numerical factors are exactly the same!) Indeed, we have shown that Eqs. (11) and (12) can be obtained, macroscopically, by the same Landau-Placzek method that leads to this  $t^{-3/2}$  behavior provided suitable assumptions are made about the effect of the long-range potential on the thermohydrodynamics of the system<sup>4</sup>; in particular, the long-range character of the Van der Waals force gives rise to potential contributions to the shear-viscosity correction.

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## Evidence for a New Phase of Solid He<sup>3</sup> †

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Measurements of the melting pressure of a sample of He<sup>3</sup> containing less than 40-ppm He<sup>4</sup> impurities, self-cooled to below 2 mK in a Pomeranchuk compression cell, indicate the existence of a new phase in solid He<sup>3</sup> below 2.7 mK of a fundamentally different nature than the anticipated antiferromagnetically ordered state. At lower temperatures, evidence of possibly a further transition is observed. We discuss these pressure measurements and supporting temperature measurements.

On the basis of measured values of the solid-He<sup>3</sup> spin exchange energy  $J$ , defined by  $\mathcal{H}_{\text{ex}} = -2J \times \sum_{i < j} I_i \cdot I_j$ , it has been assumed that near 2.0 mK solid He<sup>3</sup> would order antiferromagnetically by a second-order phase transition.<sup>1</sup> In this Letter we present evidence that at 2.7 mK solid He<sup>3</sup> undergoes a phase transition of a nature fundamentally different from that which had been expected, and that the ordered state is most probably not the simple antiferromagnetic one assumed. The refrigeration device, pressure transducer, and thermometry employed in our measurements are described, the evidence is presented, and a brief discussion follows.

The method of compressional cooling of He<sup>3</sup> to obtain temperatures as low as 2 mK is by now well established.<sup>2-4</sup> The present apparatus, shown in Fig. 1, employs a pressure amplifier which consists of a set of beryllium-copper bellows connected by a rigid piston. The pressure amplifier enables a moderate He<sup>4</sup> pressure (< 10 atm) in the upper chamber, generated externally, to compress and solidify the He<sup>3</sup> in the lower chamber. Although sufficient volume changes can be generated to solidify the entire 12-cm<sup>3</sup> He<sup>3</sup> sample, seldom was over 40% solid ever formed in the experiments to be discussed.

The apparatus was attached directly to the mixing chamber of a dilution refrigerator for pre-

cooling and thermal isolation.<sup>5</sup> Above about 5 mK the compression process was highly revers-

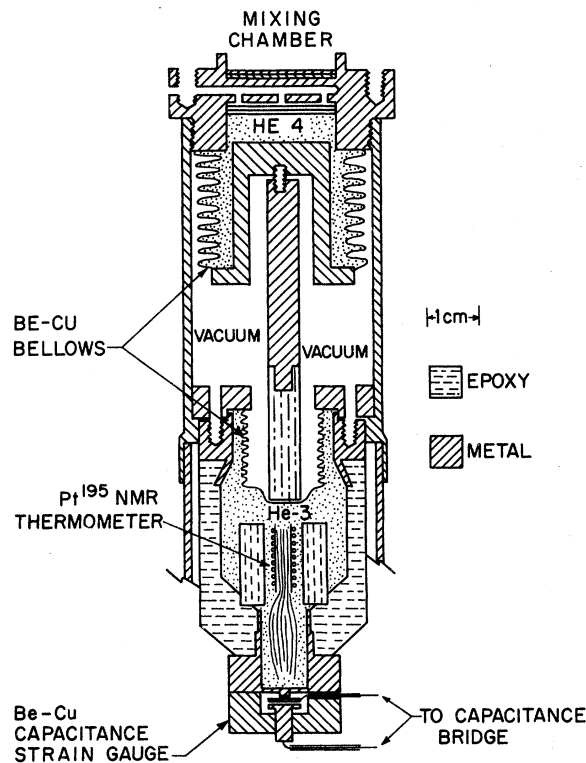


FIG. 1. Pomeranchuk cooling and pressure-measuring apparatus.