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Internal Symmetries and Model-Independent Relations for Inclusive Processes*

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Symmetries restrict isospin and SU(3) dependence of single-particle and multiparticle inclusive and semi-inclusive cross sections. Isospin relations test treatments of experimental data, including deuteron corrections, Λ-Σ^0 separations, and resonance-background separations. They also test the isospin structure of initial states in photoproduction, electroproduction, and neutrino-production processes; and they can reveal Coulomb effects in $\bar{p}d$ or $\bar{p}\alpha$ annihilation. Applied separately to beam and target fragments, they test diffractive excitation models.

We call attention to model-independent symmetry relations for inclusive reaction cross sections obtained by straightforward application of a "maximum-complexity" theorem[1-3] to specific cases. The physical content of that theorem is analogous to the statement that an initial state that contains only s and p waves cannot produce a final-state angular distribution more complicated than $A+B\cos\theta+C\cos^2\theta$. Consider the inclusive or semi-inclusive reaction

$$A+B\to C_{IM}+X, \tag{1}$$

where C_{IM} denotes the set of states within an isospin multiplet having isospin I, eigenvalue M of J_z, and X is either everything else or *all charge states* of a given type of final state, e.g., *all* five-pion states. The maximum-complexity theorem requires the isospin dependence of the cross section σ_{IM} for the inclusive Reaction (1) to be given by a polynomial in M,

$$\sigma_{IM}=\sum_{n=0}^{2I_{AB}^{(\max)}} a_{In}M^n, \tag{2}$$

of degree equal to twice the maximum isospin $I_{AB}^{(\max)}$ in the initial state.[4] Relations between the cross sections are then obtained if the number of free parameters is less than the number of independent experimental cross sections, i.e., if $I_C>I_{AB}^{(\max)}$. For initial states involving available beams on nucleon targets, $I_{AB}^{(\max)}$ is at least 1, and relations are obtained only for states having $I\geq\frac{3}{2}$. Such isospin multiplets are available only as resonances and not as stable particles. For d or ^{4}He targets, $I_{AB}^{(\max)}$ can be as low as $\frac{1}{2}$, and relations are obtainable for inclusive single-π production as well as for resonance production. Such relations can therefore be used as consistency tests on separation of resonances from background and on unscrambling of d or ^{4}He data.

The multiplet C_{IM} need not be a single particle or resonance. It could be a multiparticle system such as an N-π or a multipion state. Then the cross sections σ_{IM} for the production of a given isospin eigenstate are not directly measurable, except for the cases of maximum and minimum charge. However, the sums of cross sections for a given value of M and all possible values of I are expressible in terms of observable cross sections as shown below. For those sums, inequalities can be obtained from Eq. (2). For example,

$$\sigma_M \equiv \sum_{I=M}^{I_{\max}} \sigma_{IM} \geq \sigma_{I'M} = \sum_{n=0}^{2I_{AB}^{(\max)}} a_{I'n} M^n, \tag{3}$$

where $I_{\max}$ is the maximum isospin obtainable for the multiparticle system and I' is any value of I between M and $I_{\max}$.

Isospin equalities.—We first apply Eq. (2) to states with $I_{AB}^{(\max)} = \frac{1}{2}$. The inclusive cross section σ_{IM} must then be a linear function of M. Some equalities which hold for reactions from the initial states Kd, $\bar{K}d$, pd, $\bar{p}d$, $K\alpha$, $\bar{K}\alpha$, $p\alpha$ and $\bar{p}\alpha$ are

$$\sigma(\pi^+X) + \sigma(\pi^-X) = 2\sigma(\pi^0X), \tag{4a}$$

$$\sigma(\Sigma^+X) + \sigma(\Sigma^-X) = 2\sigma(\Sigma^0X), \tag{4b}$$

$$\sigma(\Delta_M X) = a_0 + a_1 M. \tag{4c}$$

For example, with a $\bar{p}d$ initial state and X any π-N state, Eq. (4a) becomes

$$\sigma(\bar{p}d \to \pi^+\pi^-n) + \sigma(\bar{p}d \to \pi^-\pi^+n) + \sigma(\bar{p}d \to \pi^-\pi^0p) = 2[\sigma(\bar{p}d \to \pi^0\pi^-p) + \sigma(\bar{p}d \to \pi^0\pi^0n)]. \tag{4d}$$

This relation holds for any values of the momenta of the first and second pions.

For $I_{AB}^{(\max)} = 1$, the cross section is a quadratic function of M and equalities are obtained for $I_C \geq \frac{3}{2}$. One such equality valid for initial states KN, $\bar{K}N$, NN, $\bar{N}N$, πd, $\pi\alpha$, γd, and $\gamma\alpha$ is

$$3\sigma(\Delta^+X) + \sigma(\Delta^-X) = 3\sigma(\Delta^0X) + \sigma(\Delta^{++}X). \tag{5}$$

U-spin equalities.—The same arguments can be applied to U spin or V spin if SU(3) symmetry is assumed. As the photon is a U-spin scalar, relations with U spin are most easily obtained for photoproduction experiments on protons, for which $U_{AB}^{(\max)} = \frac{1}{2}$. The most useful relations are those for decuplet baryon production, such as

$$\sigma(\gamma p \to \Delta^0X) + \sigma(\gamma p \to \Xi^{*0}X) = 2\sigma(\gamma p \to Y^{*0}X), \tag{6a}$$

$$\sigma(\gamma p \to \{\Delta^-, Y^{*-}, \Xi^{*-}, \Omega^-\}X) = a_0 + a_1 U_z. \tag{6b}$$

For initial states with $U_{AB}^{(\max)} = 1$, such as γn, $K^\pm p$, $\bar{p}p$, or $\pi^\pm p$, the U-spin analog of Eq. (5) is[2]

$$3\sigma(Y^{*-}X) + \sigma(\Delta^-X) = 3\sigma(\Xi^{*-}X) + \sigma(\Delta^-X). \tag{6c}$$

SU(3) relations involving octet production are less useful because Λ, Σ^0, π^0, and η are not eigenstates of U spin or V spin.

Multiparticle inclusive processes.—Relations (4c) and (5) hold when the Δ is replaced by any N-π system in the $I = \frac{3}{2}$ state. However, in nonresonant πN production, the $I = \frac{3}{2}$ cross section is not directly measurable for $M = \pm\frac{1}{2}$, and equalities relating observed cross sections are not obtained. Nevertheless inequalities are obtainable because the cross sections for the $M = \pm\frac{1}{2}$ states satisfy the inequality (3):

$$\sigma_{1/2} \equiv \sigma(\pi^+nX) + \sigma(\pi^0pX) = \sigma_{3/2,1/2} + \sigma_{1/2,1/2} \geq \sigma_{3/2,1/2}, \tag{7a}$$

$$\sigma_{-1/2} \equiv \sigma(\pi^0nX) + \sigma(\pi^-pX) \geq \sigma_{3/2,-1/2}. \tag{7b}$$

The inequalities (7) can be combined with the Relations (4c) and (5) for $\sigma_{3/2,M}$ to obtain inequalities relating multiparticle cross sections. In the simplest case, $I_{AB}^{(\max)} = \frac{1}{2}$ and the cross sections $\sigma_{3/2,M}$ lie on a straight line when plotted against M, as shown in Fig. 1(a). The sums $\sigma_{\pm 1/2}$ must then lie *above* this straight line, which can be determined by the two points $\sigma_{3/2,\pm 3/2}$. Thus, for the initial states Kd, $\bar{K}d$, pd, $\bar{p}d$, $K\alpha$, $\bar{K}\alpha$, $p\alpha$, and $\bar{p}\alpha$ we obtain the inequalities

$$\sigma(\pi^+nX) + \sigma(\pi^0pX) \geq \tfrac{2}{3}\sigma(\pi^+pX) + \tfrac{1}{3}\sigma(\pi^-nX), \tag{8a}$$

$$\sigma(\pi^0nX) + \sigma(\pi^-pX) \geq \tfrac{2}{3}\sigma(\pi^-nX) + \tfrac{1}{3}\sigma(\pi^+pX). \tag{8b}$$

When $I_{AB}^{(\max)} = 1$, the curve of $\sigma_{3/2,M}$ versus M is a parabola, which is not completely determined by the two points $\sigma_{3/2}$ and $\sigma_{-3/2}$. However, inequalities are still obtainable by using the Relations (7) and noting that all cross sections must be positive. This is easily seen in the extreme case $\sigma_{-3/2} = 0$, shown in Fig. 1(b). The lowest parabola that passes through the point $\sigma_{3/2,-3/2} = 0$, and

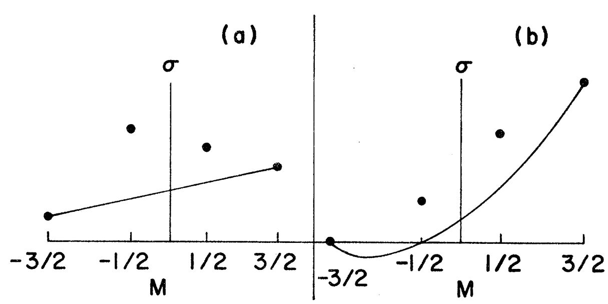

FIG. 1. Cross section versus M. (a) $I_{AB}^{(\max)} = \frac{1}{2}$; (b) $I_{AB}^{(\max)} = 1$. The cross sections at $M = \pm\frac{1}{2}$ must in each case lie above the line determined by the end points.

keeps $\sigma_{3/2,1/2} \geq 0$, passes through the point $\sigma_{3/2,-1/2} = 0$ and has the form

$$\sigma_{3/2,M} = \tfrac{1}{6}\sigma_{3/2,3/2}(M+\tfrac{3}{2})(M+\tfrac{1}{2}). \tag{9}$$

This gives the inequality

$$\sigma_{1/2} = \sigma_{3/2,1/2} + \sigma_{1/2,1/2} \geq \tfrac{1}{3}\sigma_{3/2}. \tag{10}$$

We now apply this approach to the general case where $I_{AB}^{(\max)} = 1$, for example, to reactions initiated by $K^{\pm}p$, pp, $\bar{p}p$, πd, $\pi\alpha$, γd, or $\gamma\alpha$. For the production of a π-N state, Eq. (5) with the $I=\frac{3}{2}$, π-N state in the place of Δ gives

$$\sigma_{3/2,\pm 1/2} \geq \pm\tfrac{1}{3}[\sigma_{3/2,3/2} - \sigma_{3/2,-3/2}]. \tag{11}$$

Then, the inequalities (7) lead to

$$\sigma(\pi^+nX) + \sigma(\pi^0pX) \geq \tfrac{1}{3}[\sigma(\pi^+pX) - \sigma(\pi^-nX)], \tag{12a}$$

$$\sigma(\pi^0nX) + \sigma(\pi^-pX) \geq \tfrac{1}{3}[\sigma(\pi^-nX) - \sigma(\pi^+pX)]. \tag{12b}$$

For more complicated final states, such as $2\pi + X$ or $3\pi + X$, the parabola analogous to the one displayed in Fig. 1(b) does not have to extend to the maximum values of $|M|$. Any pair $\sigma_{\bar{m}}$ and $\sigma_{-\bar{m}}$ with $\frac{3}{2} \leq \bar{m} \leq I_C^{(\max)}$ can be used to define a partial cross section

$$\sigma_M^{\bar{m}} \equiv \sum_{I=\bar{m}}^{I_C^{(\max)}} \sigma_{IM} \leq \sigma_M. \tag{13}$$

For $I_{AB}^{(\max)} = 1$, the maximum-complexity theorem requires that $\sigma_M^{\bar{m}}$ is a quadratic function of M for $|M| \leq \bar{m}$. The two endpoints of the parabola $\sigma_M^{\bar{m}}$ are determined by the experimental cross sections $\sigma_{\pm\bar{m}}^{\bar{m}} = \sigma_{\pm\bar{m}}$. Experimental cross sections cannot completely determine the parabola, but the requirement that $\sigma_M^{\bar{m}} \geq 0$ for allowed values of M such that $|M| < \bar{m}$ defines $2\bar{m}-1$ parabolas each of which lies below the parabola defined by $\sigma_m^{\bar{m}}$. Then the inequality (13) gives

$$\sigma_M(I_{AB}^{(\max)} = 1) \geq \sigma_M^{\bar{m}} \geq \frac{M-m}{2\bar{m}}\left(\frac{\bar{m}+M}{\bar{m}-m}\,\sigma_{\bar{m}} - \frac{\bar{m}-M}{\bar{m}+m}\,\sigma_{-\bar{m}}\right) \tag{14}$$

for all allowed M and m such that $|M| < \bar{m}$ and $|m| < \bar{m}$. The choice of m and $\bar{m}$ to give the best inequality (14) for a particular M value depends upon the experimental cross sections $\sigma_{\pm\bar{m}}$. Thus, nonisospin criteria, such as momentum selection, may be used to strengthen the inequality.

Setting $\bar{m} = \frac{3}{2}$ in inequality (14) gives inequality (13). For the next case, $\bar{m} = 2$, we find

$$\sigma_M(I_{AB}^{(\max)} = 1) \geq \frac{M-m}{4}\left(\frac{2+M}{2-m}\,\sigma_2 - \frac{2-M}{2+m}\,\sigma_{-2}\right), \tag{15}$$

where $|M|$, $|m| < 2$. This relation applies, for instance, to multipion inclusive processes, such as the two-π reactions

$$A + B \to \pi(k_1) + \pi(k_2) + X, \tag{16}$$

where AB could be $K^{\pm}p$, pp, $\bar{p}p$, πd, or γd and the momenta k_1 and k_2 are specified to distinguish between the two π's. The choice $|\bar{m}| = 2$ is convenient experimentally since the doubly charged states are most easily identified. Substituting M values into (15) and trying $m = 0, \pm 1$ then gives for Reactions (16)

$$\sigma_{\pm 1}(I_{AB}^{(\max)} = 1) \geq \tfrac{1}{2}\sigma_{\pm 2} - \tfrac{1}{2}\sigma_{\mp 2}, \tag{17a}$$

$$\sigma_{\pm 1}(I_{AB}^{(\max)} = 1) \geq \tfrac{3}{8}\sigma_{\pm 2} - \tfrac{1}{8}\sigma_{\mp 2}, \tag{17b}$$

$$\sigma_0(I_{AB}^{(\max)} = 1) \geq \tfrac{1}{6}\sigma_{\pm 2} - \tfrac{1}{4}\sigma_{\mp 2}, \tag{17c}$$

where for the two-pion Reactions (16)

$$\sigma_{\pm 2} \equiv \sigma(\pi^{\pm}\pi^{\pm}X), \quad \sigma_{\pm 1} \equiv \sigma(\pi^{\pm}\pi^0X) + \sigma(\pi^0\pi^{\pm}X), \tag{18a}$$

$$\sigma_0 \equiv \sigma(\pi^+\pi^-X) + \sigma(\pi^-\pi^+X) + \sigma(\pi^0\pi^0X). \tag{18b}$$

Inequalities (17) also hold for multipion inclusive processes, provided that σ_M includes all n-pion states of charge M. Additional relations are obtained by n-pion states by using $m = 3, 4, \cdots, n$ in inequality (15) and trying the available values of m.

All the relations derived in this Letter hold for any set of fixed values of the momenta of the outgoing particles and can be tested at each point in an experimental energy spectrum or angular distribution. Isospin relations can test consistency of experimental data analysis, including separation of resonances from background and treatment of deuteron data. The equality (4b) can be used to separate Σ^0 and Λ production.[3] Inequalities (12) and (17) applied to γd or $\gamma\alpha$ reactions could test for an isotensor component in the electromagnetic current.[5] Applications to neutrino production are discussed elsewhere.[6] The equalities (4) applied to $\bar{p}d$ annihilation at rest could test for Coulomb effects in the capture and annihilation process as a function of the momentum of the final nucleon.

The validity of SU(3) symmetry for inclusive reactions might be questioned in view of obviously large kinematic symmetry-breaking effects which make multipion final states more frequent than multikaon states. However, the same arguments apply to total cross sections where relations following from SU(3) and the optical theorem are satisfied to a surprising degree. The experimental tests of the U-spin relations (6) are therefore of interest.

These model-independent relations follow from isospin and U-spin invariance, respectively, and will hold in any model (e.g., in the Mueller-Regge model) if the model does not violate isospin or SU(3) symmetry. Additional model-dependent symmetry relations have been obtained from particular models.[7] Those usually follow from assumptions that limit the quantum numbers in a particular channel to those of allowed (nonexotic) Regge trajectories, or to be those of the Pomeranchukon in the case of a diffractive process.

The isospin relations can be used to test diffractive excitation models which assume that there is no isospin exchange between the beam and target.[8] The relations can be applied separately in the beam and target fragmentation regions by setting $I_{AB}^{(\max)}$ equal to the beam and target isospin respectively. For example, in the fragmentation region of a nucleon or kaon, Eq. (4a) should hold at any value of the pion momentum. It would be interesting to test these models at CERN intersecting-storage-ring energies by examining the inclusive π^0 production in a kinematic region where the π^+ and π^- cross sections are very different from one another.

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[4]To prove (2), express σ_{IM} as a sum of irreducible tensor polynomials $P_L(I, M)$, analogous to the spherical harmonics Y_{LO}. The Wigner-Eckart theorem implies that the maximum L in the sum is not greater than $2I_{AB}^{(\max)}$. Details are given by Peshkin (Ref. 1). [Eq. (2.16) should read $Q_2 = \frac{3}{2}J_z^2 - \frac{1}{2}J(J+1)$].

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Isospin Constraints on Semi-inclusive Neutrino Reactions and Their Hadronic Analogs*

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The processes $\nu(\bar{\nu})$ + target → hadron + $\mu(\bar{\mu})$ + anything are considered in the cases where the hadron selected has $I = \frac{1}{2}$ or 1. Inequalities are given which follow from the conventional assignment of isospin to the weak current (for $\Delta S = 0, 1$) and the charge-symmetry conditions for the $\Delta S = 0$ current. νd and $\bar{\nu} d$ processes with $\Delta S = 1$ yield an equality if the hadron selected has $I = 1$. The hadronic analogs of these reactions are also discussed; bounds are given for $\pi^- + p$ → hadron + anything, which are stronger than those previously reported.

In this note we record bounds on the neutrino-induced cross sections for producing neutral hadrons with $I \neq 0$ in terms of the production cross sections for their charged counterparts. These bounds may provide a useful constraint on the "missing-neutrals" problem which plagues all neutrino experiments.