Gravitational Radiation Reaction*

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We present the results of an exact calculation of the equations of motion (with gravitational radiation reaction terms) of a gravitating system subject to no external forces.

The purpose of this note is to present the results of an exact calculation of the equations of motion (which naturally contain gravitational radiation reaction terms) of a gravitating system subject to no external forces. The novelty of our approach lies in the fact that the system is to be considered as the source of an asymptotically flat space and all the relevant physical quantities such as four-velocity v^μ, four-momentum p^μ, angular-momentum center-of-mass tensor $S^{\mu\nu}$, as well as the higher mass and rotational moments are then *defined* in terms of surface integrals taken at infinity,[1,2] analogous to *defining* charge by Gauss's theorem. A subset of the Einstein equations (equivalent to Bondi's supplementary conditions[3,4]) then yields the following equations for these quantities (analogous to showing charge conservation from Maxwell's equations):

$$\dot{S}^{\mu\nu}+v^{[\mu}p^{\nu]}=J^{\mu\nu}-\tfrac{1}{3}Tv^{[\mu}\dot{v}^{\nu]}-\tfrac{1}{2}t^{[\mu}\dot{v}^{\nu]}+\tfrac{1}{3}t^{\alpha[\mu}{}_{v}{}^{\nu]}\dot{v}_\alpha, \tag{1}$$

$$\dot{p}^\mu=F^\mu+\tfrac{1}{6}T\dot{v}^\alpha\dot{v}_\alpha v^\mu-\tfrac{1}{3}\dot{T}\dot{v}^\mu+\tfrac{1}{2}\dot{t}^{\mu\alpha\beta}v_\alpha\dot{v}_\beta+2t^{\mu\alpha\beta}\dot{v}_\alpha\dot{v}_\beta. \tag{2}$$

$J^{\mu\nu}$ and F^μ are, respectively, the radiation reaction torque and force due to the mass (sometimes called "electric-type") moments and spin ("magnetic-type") moments. Because of their length and because their details play no role here, their precise form is omitted. The quantities T, t^μ, $t^{\mu\nu}$, and $t^{\mu\nu\rho}$ are all defined from a trace-free symmetric tensor $T^{\mu\nu\rho}$ by first writing

$$T^{\mu\nu}=\tfrac{3}{4}T^{\mu\nu\rho}v_\rho,\quad T^\mu=\tfrac{2}{3}T^{\mu\nu}v_\nu=\tfrac{1}{2}T^{\mu\nu\rho}v_\nu v_\rho,\quad T=\tfrac{1}{2}T^\mu v_\mu=\tfrac{1}{3}T^{\mu\nu}v_\mu v_\nu=\tfrac{1}{4}T^{\mu\nu\rho}v_\mu v_\nu v_\rho, \tag{3}$$

and then

$$t^{\mu\nu\rho}=T^{\mu\nu\rho}-T(v^\mu v^\nu v^\rho-\eta^{(\mu\nu}v^{\rho)}),\quad t^{\mu\nu}=T^{\mu\nu}-T(v^\mu v^\nu-\tfrac{1}{2}\eta^{\mu\nu}),\quad t^\mu=T^\mu-Tv^\mu,\quad t^\mu v_\mu=0. \tag{4}$$

(For reasons of convention the v^μ has been normalized so that $v_\mu v^\mu=2$.)

$T^{\mu\nu\rho}$ is defined from the asymptotic shear σ^0, by the Lorentz invariant integral

$$T^{\mu\nu\rho}l_\mu l_\nu l_\rho=\int K(\Omega,\Omega')(\sigma^0\bar{\sigma}^0/v^5)\,d\Omega', \tag{5}$$

Ω and Ω' being points on the unit sphere, $d\Omega$ the element of area, with

$$K(\Omega,\Omega')=Y_{00}(\Omega)\bar{Y}_{00}(\Omega')-\tfrac{3}{5}\textstyle\sum_m Y_{1m}(\Omega)\bar{Y}_{1m}(\Omega')+\tfrac{1}{5}\sum_m Y_{2m}(\Omega)\bar{Y}_{2m}(\Omega')-\tfrac{1}{35}\sum_m Y_{3m}(\Omega)\bar{Y}_{3m}(\Omega'), \tag{6}$$

$$l_\mu=(2\sqrt{2}\,P_0)^{-1}(1+\zeta\bar{\zeta},\ \zeta+\bar{\zeta},\ (\zeta-\bar{\zeta})/i,\ -1+\zeta\bar{\zeta}), \tag{7}$$

$$v=v^\mu l_\mu. \tag{8}$$

Integrals and kernals of this type arise frequently in the theory of representations of the homogeneous Lorentz group.[5,6] This theory has played a dominant role in the work presented here.

We do not give here the definitions of v^μ, p^μ, and $S^{\mu\nu}$, nor the justification for these definitions other than to say that they reduce in the linear theory to the usual definitions and that they transform in the full theory properly under the homogeneous Lorentz group which is well defined in asymptotically flat spaces.

From (1) by contracting with v^μ and defining the inertial mass by

$$m=p^\mu v_\mu, \tag{9}$$

we obtain

$$p^{\mu} = mv^{\mu} + \dot{S}^{\mu\alpha}v_{\alpha} - J^{\mu\alpha}v_{\alpha} - \tfrac{1}{3}T\dot{v}^{\mu} + \tfrac{1}{4}t^{\alpha}\dot{v}_{\alpha}v^{\mu} - \tfrac{1}{3}t^{\mu\alpha}\dot{v}_{\alpha}. \tag{10}$$

Substituting (10) into (2) yields

$$m\dot{v}^{\mu} = F^{\mu} - \tfrac{1}{2}F^{\alpha}v_{\alpha}v^{\mu} - \ddot{S}^{\mu\alpha}v_{\alpha} + \dot{J}^{\mu\alpha}v_{\alpha} + \tfrac{1}{3}T(\ddot{v}^{\mu} + \tfrac{1}{2}\dot{v}^{\alpha}\dot{v}_{\alpha}v^{\mu}) + \tfrac{1}{4}t^{\mu}\dot{v}^{\alpha}v_{\alpha} - \tfrac{1}{2}T^{\alpha}\dot{v}_{\alpha}\dot{v}^{\mu} + \tfrac{3}{4}(\dot{t}^{\mu\alpha\beta} - \tfrac{1}{2}\dot{t}^{\alpha\beta\gamma}v_{\gamma}v^{\mu})v_{\alpha}\dot{v}_{\beta} + \tfrac{1}{4}(t^{\mu\alpha\beta} - \tfrac{1}{2}t^{\alpha\beta\gamma}v_{\gamma}v^{\mu})(v_{\alpha}\ddot{v}_{\beta} + 9\dot{v}_{\alpha}\dot{v}_{\beta}) \tag{11}$$

and

$$\dot{m} = \tfrac{1}{2}(F^{\alpha}v_{\alpha} - \dot{S}^{\alpha\beta}v_{\alpha}\dot{v}_{\beta} + J^{\alpha\beta}v_{\alpha}\dot{v}_{\beta} + \dot{t}^{\alpha}\dot{v}_{\alpha} + t^{\alpha\beta}\dot{v}_{\alpha}\dot{v}_{\beta}). \tag{12}$$

It should be noticed that in (11), though the fifth term is similar to the radiation reaction term in the Lorentz-Dirac force law with $\frac{2}{3}e^2$ replaced by $\frac{1}{3}T$ (T can be shown to be non-negative), the radiation reaction force is vastly more complicated in the gravitational case.

Though (12) yields the time development of the inertial mass, it is the p^0 which is the Bondi mass and (2) yields the Bondi mass loss, i.e., it can be shown that $\dot{p}^0 \leq 0$.

In the units used here the gravitational constant is 1. When conventional units are used and the limit of zero gravitational constant is taken, (1) and (2) reduce to

$$S^{\mu\nu} + v^{[\mu}p^{\nu]} = 0, \quad \dot{p}^{\mu} = 0, \tag{13}$$

the usual[7] Lorentz-invariant equations of motion of a particle with intrinsic angular momentum.

In addition to the question of the reasonableness of our definitions of the physical quantities, momentum etc., there is the serious and difficult question of which asymptotic coordinate system is to be used in order to evaluate these quantities. The evidence seems to point to a unique canonical[8] choice which would lead to the further condition

$$S^{\mu\nu}v_{\nu} = 0, \tag{14}$$

i.e., to center-of-mass coordinates. It must be admitted that the proof of this supposition is very difficult, and for the time being we would be satisfied with its proof in the linearized theory.

The details of this work will be presented in two papers being prepared for publication.

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