are the coefficients to be determined using Eq. (3) and the data. Using the method of least squares, we find that the dispersion in the sound velocity is given by $\delta(\nu)=(6.3\pm0.7)\times10^{-13}\nu+(0\pm8)\times10^{-25}\nu^2$, with ν given in units of hertz. The values from Molinari and Regge[5] in these units are $\delta(\nu)=7.3\times10^{-13}\nu-4.7\times10^{-24}\nu^2$. The agreement in the linear term is rather remarkable, while not much can be said about the small quadratic term. The data are not consistent with an expansion for the dispersion of $\delta(\nu)=A\nu^2+B\nu^4$. Thus, the experimental results give direct evidence that the dispersion is positive and that the linear term must be included in the expansion.

The values for the coefficients C_1 and C_2 for the phase shift are $C_1=(-0.12\pm0.01)\times10^{-9}$ and $C_2=(6\pm1)\times10^{-22}$. Using these coefficients the relative values of the phase shift can be obtained, while the absolute scale is obtained by a simple linear extrapolation to zero thickness d of data similar to that given in Fig. 1. A phase shift of 3.6 Å is obtained at 20 GHz, and a value of 0.8 Å at 60 GHz. The very rapid decrease in the phase shift suggests that the energy of the elementary excitations in helium is suppressed in the vicinity of the liquid-solid interface where the helium is under large pressures. More evidence for the deformation of the phonon spectrum has recently been given by Jäckle and Kehr.[11] If this is the case, this may be a natural explanation for the Kapitza-resistance dilemma.

We have been informed by Regge[12] that the following information was inadvertently dropped from the paper of Molinari and Regge.[5] The coefficients given in Eq. (3) of their paper were incorrectly rounded off. The correct fit is

$$\epsilon(p)=C\hbar p(1+0.5465p-1.3529p^2+0.2595p^3+0.1860p^4-0.0522p^5)^{1/2}.$$

These coefficients may change by as much as 30% within a standard deviation. However, the corresponding errors are very strongly correlated, and independent rounding off is not allowed.

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Stationary States of Two-Dimensional Turbulence

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A class of exact statistical equilibria for the Navier-Stokes equation not including viscosity in two dimensions is noted and the corresponding fluctuation spectrum calculated. These solutions may account for the phenomena recently observed in numerical simulations of two-dimensional turbulence by Deem and Zabusky, including the existence of two distinct regimes of turbulence and the relaxation of one of these towards equipartition of vorticity.

There is considerable interest in two-dimensional hydrodynamic turbulence, partly because it may be relevant to meteorology and oceanography, but mainly because it is amenable to investigation by high-resolution numerical methods.[1-3]

Deem and Zabusky[1] have recently reported such numerical calculations and have interpreted their results in terms of the existence of two regimes. In the first regime (case 1) the equilibrium scalar-model energy spectrum takes the form

$$E(k)\propto k^{-\mu} \tag{1}$$

with a value of μ close to 4. The spectrum $E(k)$ is related to the fluid energy by

$$\tfrac{1}{2}\langle u^2\rangle = \int_0^\infty E(k)\,dk. \qquad (2)$$

This result is consistent with an argument of Chorin[4] and the theoretical prediction ($\mu = 4$) by Saffman[5] for the inertial-range spectrum. It differs somewhat from the theories of Kraichnan,[6] Leith,[7] and Batchelor,[8] who predict a value of $\mu = 3$. In the second regime (case 2) Deem and Zabusky find a completely different behavior. The initial spectrum evolves into one which at large wave numbers fits closely to a k^{-1} law; and as the calculation proceeds, this k^{-1} spectrum extends towards ever lower k.

The vorticity representation of the two-dimensional (2D) Navier-Stokes equation not including viscosity, which governs the inertial range of turbulence, is formally identical with the equations describing a 2D guiding-center plasma. This has recently been investigated by Taylor and Thompson,[9] who developed a kinetic equation for a guiding-center plasma and deduced that the equilibrium spectrum has the form[9]

$$E(k) \propto k/(a^2+k^2). \qquad (3)$$

We would like to point out that there is indeed a family of *exact* statistical equilibria given by the Navier-Stokes equation for an inviscid fluid, that these have a spectral distribution of the form (3), and that the computational results of Deem and Zabusky can be understood in the light of these exact solutions.

The motion of an incompressible, inviscid fluid in the x, y plane is described in terms of the vorticity $\omega = \hat{z}\cdot\nabla\times\vec{u}$ by the equations[10]

$$\partial\omega/\partial t + [\psi, \omega] = 0, \quad \nabla^2\psi = -\omega, \qquad (4)$$

where ψ is the stream function. If the vorticity be expanded in Fourier series over a unit square,

$$\omega(\vec{r}, t) = \textstyle\sum_k \rho_k(t) e^{i\vec{k}\cdot\vec{r}}, \qquad (5)$$

then the ρ_k satisfy the equation

$$\frac{\partial\rho_k}{\partial t} = \sum_{j,l} M_{jl}^{-k}\rho_j\rho_l, \qquad (6)$$

where

$$M_{jl}^{k} \equiv \tfrac{1}{2}\hat{z}\cdot(\vec{j}\times\vec{l})(l^{-2}-j^{-2})\delta_{kjl}, \qquad (7)$$

$$\delta_{kjl} \equiv \begin{cases} 1 \text{ if } \vec{k}+\vec{j}+\vec{l}=0, \\ 0 \text{ if } \vec{k}+\vec{j}+\vec{l}\neq 0. \end{cases} \qquad (8)$$

The spectral function discussed by Deem and Zabusky is

$$E(k) = (2\pi/k)\langle\rho_k\rho_{-k}\rangle, \qquad (9)$$

the angular brackets denoting an ensemble average.

The following properties of the coefficients M_{jl}^{k} are important;

$$k^{-2}M_{jl}^{k} + j^{-2}M_{lk}^{j} + l^{-2}M_{kj}^{l} = 0, \qquad (10)$$

$$M_{jl}^{k} + M_{lk}^{j} + M_{kj}^{l} = 0. \qquad (11)$$

Equations (10) and (11) ensure the conservation of energy and of squared vorticity (enstrophy), respectively.

We now introduce the characteristic functional

$$G(\{\mu_k\}, t) \equiv \langle\exp[i\textstyle\sum\mu_k\rho_k(t)]\rangle; \qquad (12)$$

which contains all the statistical information about the ensemble and whose derivatives generate all the moments of the ρ_k; in particular,

$$\langle\rho_k\rho_{-k}\rangle = -[\partial^2 G/\partial\mu_k\partial\mu_{-k}]_{\{\mu_k\}=0}.$$

Then G satisfies a linear equation, similar to Liouville's equation,

$$i\frac{\partial G}{\partial t} = \sum_{k,j,l} M_{jl}^{k}\mu_{-k}\left[\frac{\partial^2 G}{\partial\mu_j\partial\mu_k}\right]. \qquad (13)$$

We seek stationary solutions of this equation in which G is of the form

$$G = G(-\tfrac{1}{2}\textstyle\sum\mu_k\mu_{-k}Q_k) \qquad (14)$$

for which the covariance $\langle\rho_k\rho_{-k}\rangle \propto Q_k$. It is easily verified that (14) is a stationary solution of (13) provided Q_k satisfies (for all k, j, l)

$$\frac{1}{Q_k}M_{jl}^{k} + \frac{1}{Q_j}M_{lk}^{j} + \frac{1}{Q_l}M_{kj}^{l} = 0. \qquad (15)$$

Using (10) and (11) it is clear that this constraint is satisfied if

$$Q_k \propto k^2/(a^2+k^2). \qquad (16)$$

[Indeed it can be shown[9] that this is the only analytic solution of (15).] Thus, for any value of a^2 the spectrum (16) for Q_k, and therefore the energy spectrum (3), corresponds to an *exact* statistical equilibrium for the inertial-range equations.

We turn now to the interpretation of the numerical calculations of Deem and Zabusky. We suggest that the spectrum towards which their system evolves in the second regime (case 2) is simply the exact stationary spectrum described above, and is given by Eq. (3). For the particular parameters used by Deem and Zabusky, the parameter a^2 will be very small (see below) so that the spectrum closely approximates a k^{-1} power law as they observed.

This small value of a^2 in case 2 and the failure of case 1 to show a similar relaxation are both accounted for by the following considerations. As we have already noted, the energy

$$\tfrac{1}{2}\langle u^2\rangle = \int_0^\infty E(k)\,dk \qquad (17)$$

and the enstrophy

$$\tfrac{1}{2}\langle \omega^2\rangle = \int_0^\infty k^2 E(k)\,dk \qquad (18)$$

are both exact constants of motion for an inviscid fluid. Furthermore, they vary slowly in the numerical experiments (cf. table II of Ref. 1). Thus, an initial state can relax rapidly toward the equilibrium (3) only if this equilibrium and the initial state have similar values for $\langle\omega^2\rangle$ and $\langle u^2\rangle$. As the spectra have an arbitrary scale this requirement is met if $\langle\omega^2\rangle/\langle u^2\rangle$ is the same in the initial and equilibrium states; this condition selects the appropriate value of a^2 in Eq. (3).

It is important to note, therefore, that when the permitted wave numbers are limited (as in numerical calculations in a periodic box), *all* spectra of the form (3) yield values of $\langle\omega^2\rangle/\langle u^2\rangle$ lying within a finite range. The limits are set by the values for $a=0$ and $a=\infty$ and[11]

$$\frac{k_1^2 - k_0^2}{2\ln(k_1/k_0)} \le \frac{\langle\omega^2\rangle}{\langle u^2\rangle} \le \tfrac{1}{2}(k_1^2 + k_0^2), \qquad (19)$$

where k_0 and k_1 are the minimum and maximum wave numbers. With $k_0 = 1$ and $k_1 = 64$ (in units of $2\pi/L$), as in the computations of Deem and Zabusky, the limits are $500(2\pi/L)^2$ and $2000(2\pi/L)^2$, and for complete relaxation to be possible the initial value of $\langle\omega^2\rangle/\langle u^2\rangle \equiv m_0$ must be within this range.

The initial spectra are given by Deem and Zabusky, so we can determine m_0 for each of their two situations; for case 1, $m_0 = 10(2\pi/L)^2$ and for case 2, $m_0 = 200(2\pi/L)^2$. We see therefore that case 2 is close to the regime permitting complete relaxation to the equilibrium spectrum (3). One could expect it to relax to a spectrum close to (3) with $a^2 \sim 0$ and with rather more than the correct energy in the lower modes—as is in fact observed. On the other hand, the value of m_0 for case 1 is so far removed from that permitting full relaxation that no significant relaxation is possible.

In conclusion then, we have noted the existence of exact statistical equilibria for the 2D Navier-Stokes equation not including viscosity (corresponding to those of a guiding-center plasma). These steady states appear to account well for the phenomena observed by Deem and Zabusky in their numerical calculations of 2D turbulence including the qualitatively different behavior of cases 1 and 2 and the observed energy spectrum in the latter.

There remains the question of the relation of these exact statistical equilibria to the pseudo-equilibria discussed in Refs. 4–8. The latter must correspond to situations with small but finite viscosity and a source of vorticity and energy. In these circumstances the steady state may be of the cascade type with energy and vorticity flowing from one part of the spectrum to another. A similar situation occurs in 3D turbulence where Hopf and Titt[12] have shown that the only Gaussian stationary state of the theory for an inviscid fluid has a spectral distribution function corresponding to equipartition of energy—a result in striking contrast to the Kolmogorov theory. Our result is analogous to that of Hopf and Titt. If $a=0$, it corresponds to equipartition of vorticity; if $a=\infty$, it corresponds to equipartition of energy.

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