from those arising through theoretical uncertainty in the short-range part. In this regard, the situation hopefully may turn out to be somewhat similar to that encountered in the analysis of nucleon-nucleon scattering; despite the fact that the low partial waves are treated phenomenologically, one is able to tie down the value of the coupling constant entering the long-range part (the Yukawa potential, arising from one-pion exchange) by using it to compute the contribution from high partial waves.

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Numerical Simulation of Three-Dimensional Homogeneous Isotropic Turbulence

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This Letter reports numerical simulations of three-dimensional homogeneous isotropic turbulence at wind-tunnel Reynolds numbers. The results of the simulations are compared with the predictions of the direct-interaction turbulence theory.

There has been much pessimism concerning the prospects for numerical simulation of three-dimensional turbulent flows.[1] This pessimism seems well founded for the accurate numerical simulation of huge-Reynolds-number flows, but our results[2] show that numerical simulation is feasible and even economical at Reynolds numbers like those achieved in wind-tunnel turbulence experiments. A number of simulation runs have been made with microscale Reynolds numbers[3] in the range $R_\lambda \leq 45$. In this Letter, we emphasize three runs made at $R_\lambda = 35$. We believe that the value of our simulations lies not only in the completeness of the data they provide, but also in the opportunity they give for the assessment of the accuracy of turbulence theories under controlled (and known) conditions.

The three-dimensional Navier-Stokes equations for incompressible flow are solved numerically using a Galerkin approximation based on a Four-

ier-series representation of the flow field $\vec{v}(\vec{x}, t)$.[4] In order to facilitate comparison with existing calculations of the direct-interaction theory,[5] periodic boundary conditions are imposed on the sides of a cube box of side length π, so that $\vec{v}(\vec{x}, t)$ has the discrete Fourier representation

$$\vec{v}(\vec{x}, t) = \sum \vec{u}(2\vec{k}, t) e^{2i\vec{k}\cdot\vec{x}}, \tag{1}$$

where $\vec{k}$ has integral components and the notation is selected to emphasize that only even Fourier components appear in (1). An approximate solution to the Navier-Stokes equations is sought in terms of a truncated series of the form (1) with $|2\vec{k}| < K$, where K is a cutoff wave number. When the truncated expansion (1) is used in the Navier-Stokes equations, there results the Galerkin equations

$$(\partial/\partial t + 4\nu k^2) u_\alpha(2\vec{k}, t) = -2ik_\beta(\delta_{\alpha\gamma} - k_\alpha k_\gamma/k^2) \sum{}' u_\beta(2\vec{p}, t) u_\gamma(2\vec{k} - 2\vec{p}, t), \tag{2}$$

where ν is the kinematic viscosity; the summation convention is implied; Greek subscripts range on 1, 2, 3; and $\sum'$ denotes a sum over all wave vectors $2\vec{p}$ with even integral components satisfying the truncation conditions $|2\vec{p}| < K$ and $|2\vec{k} - 2\vec{p}| < K$. Equations (2) embody the incompressibility condition in the sense that if $k_\alpha u_\alpha(\vec{k}, 0) = 0$ for all retained $\vec{k}$, then $k_\alpha u_\alpha(\vec{k}, t) = 0$ for all $t > 0$. The finite set of ordinary differential equations in t given by (2) for $|2\vec{k}| < K$ is solved by leapfrog time differencing on the nonlinear terms and by Crank-Nicolson implicit time differencing on the viscous terms.[6]

Judicious application of the fast Fourier-transform algorithm[7] using finite analogs of the convolution theorem[8] permits great efficiencies in the implementation of Eqs. (2). The simulations reported here are obtained using the cutoff $K = 2(242)^{1/2} \simeq 31.1$. It has been estimated[9] that the cutoff-31.1 simulations are at least as accurate as finite-difference simulations using $(64)^3$ grid points (786 432 velocity values) within the cubic box of side π. Our code requires about 30 sec per time step on a CDC 6600 computer; typical decay calculations involve 200 time steps or about $1\frac{1}{2}$ h of CDC 6600 computer time.

The initial flow field is chosen as an incompressible realization of a zero-mean Gaussian ensemble with isotropic energy spectrum[10] of the form $E(k) = Ak^4 \exp(-Bk^2)$. Here A and B are chosen so that $v_{\mathrm{rms}}(0) = 1$, where the rms velocity $v_{\mathrm{rms}}(t)$ is defined so that $\frac{3}{2} v_{\mathrm{rms}}^2(t)$ is the average kinetic energy per unit volume at time t, and the maximum of $E(k)$ occurs at $k_{\max} = 2^{9/4} \simeq 4.756\,83$.[5] The particular runs reported in detail here are labeled $3A$, $3B$, and $3C$, each a different realization of a random flow field with a common set of run-3 parameter values. For runs 3, $\nu = 0.011\,89$, $\Delta t = 0.004$, $L(0) = 0.54$, $\lambda(0) = 0.42$, and $R_\lambda(0) = 35.4$, where Δt is the (constant) time step, $L(t)$ is the longitudinal integral scale[10] at t, $\lambda(t)$ is the Taylor microscale at t, and $R_\lambda(t) = v_{\mathrm{rms}}(t)\lambda(t)/\nu$.

At specified times, averages are computed as $1/\Delta k$ times the sum over wave vectors lying within shells in Fourier space centered at odd-integer wave numbers k_0 and of width Δk. We take $\Delta k = 2$. The two-time energy $\bar{E}(k_0; t, t')$, dissipation $\bar{D}(k_0, t)$, and transfer $\bar{T}(k_0, t)$ spectra are thus defined as band averages of $\frac{1}{2} u_\alpha(\vec{k}, t) u_\alpha(-\vec{k}, t)$, $\nu k^2 |\vec{u}(\vec{k}, t)|^2$, and $r_\alpha(\vec{k}, t) u_\alpha(-\vec{k}, t)$, respectively, where $r_\alpha(2\vec{k}, t)$ denotes the right-hand side of (2). We also compute band-averaged error spectra, defined as the rms expected error in the Fourier-band averages if fluctuations are statistically independent.

The band-averaged time-correlation function, defined by

$$\bar{R}(k; t, t_0) = \frac{\bar{E}(k; t, t_0)}{[\bar{E}(k; t, t)\bar{E}(k; t_0, t_0)]^{1/2}},$$

is plotted in Fig. 1 for run $3C$. The abscissa is chosen so that, if the dominant effect causing decay of $\bar{R}$ to zero for large k is the decorrelation of small-scale (high-k) structures by convection on large-scale energy-containing motions, then all the data should collapse to a single curve. Evidently, convection of small scales by large ones is an important factor in the behavior of $\bar{R}$.

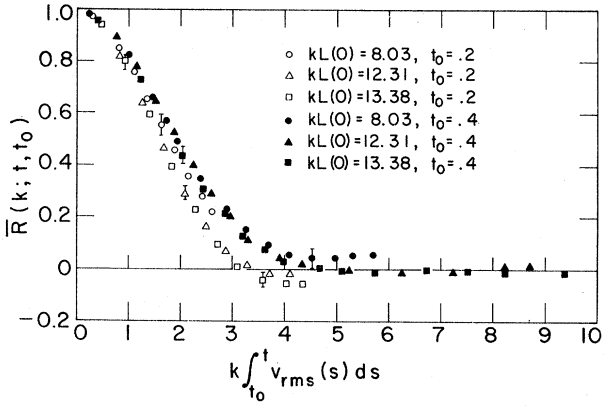

FIG. 1. Time-correlation function for run $3C$.

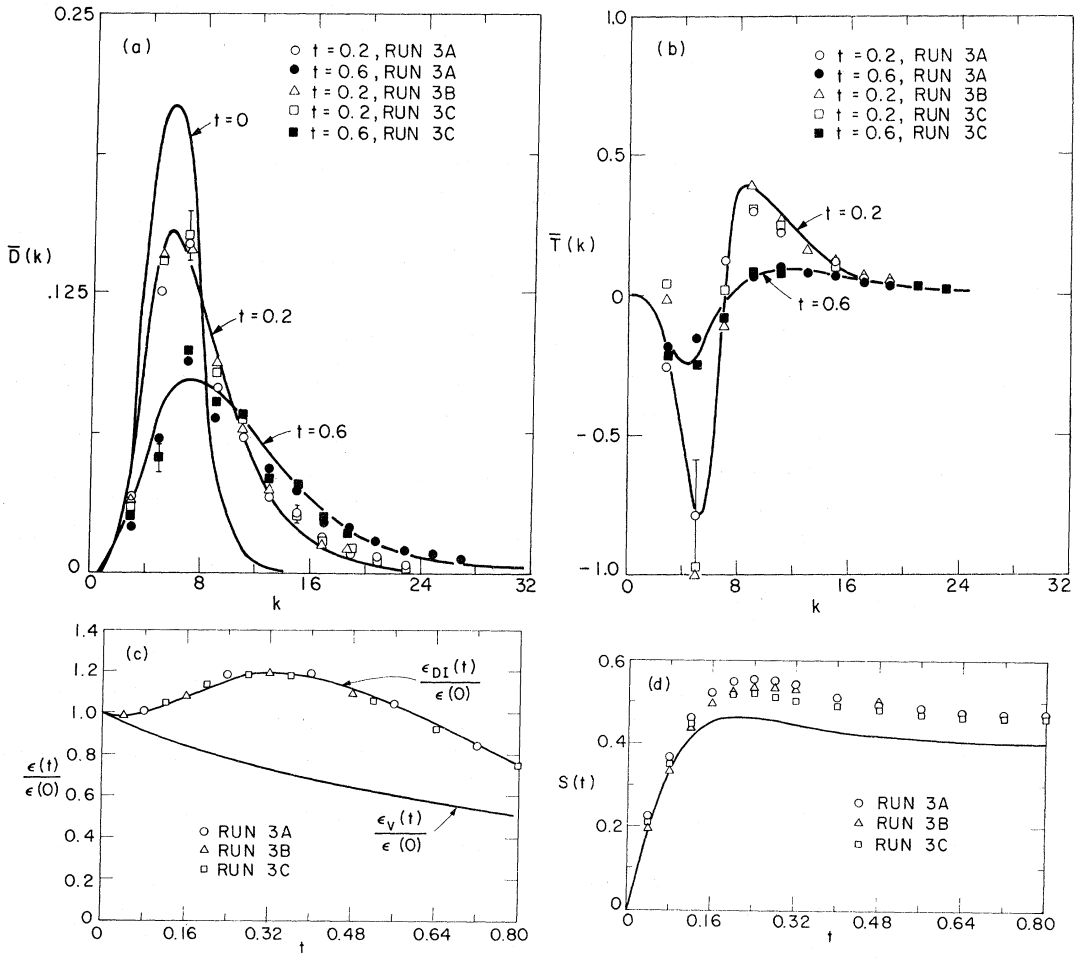

FIG. 2. Comparison between numerical simulations (data points) and direct-interaction (DI) theory (solid curves) for run 3. (a) Dissipation spectrum, (b) transfer spectrum, and (c) energy dissipation rate. The curve labeled ϵ_v is for pure viscous decay, obtained by dropping the right-hand side of (2). (d) Skewness.

The decay of $\overline{R}$ to 0 (within statistical fluctuations) is proof that we are simulating turbulence and not a laminar flow with persistent correlations.

In Figs. 2(a)-2(d), we compare the results of runs $3A$–$3C$ with calculations of the direct-interaction approximation using the same initial conditions.[5] The direct-interaction approximation does a good job of predicting the evolution of the sensitive spectra $\overline{D}$ and $\overline{T}$ and the energy dissipation rate $\epsilon(t) = -d[\frac{3}{2}v_{\mathrm{rms}}{}^2(t)]/dt$.

The skewness[10] of the longitudinal velocity derivative is conveniently expressed as

$$S(t) = \tfrac{2}{35}[\lambda(t)/v_{\mathrm{rms}}(t)]^3 \int k^2 T(k, t)\, dk.$$

$S(t)$ is a dimensionless measure of the production of vorticity by turbulent cascade of energy. The comparison with the direct-interaction results for $S(t)$ in Fig. 2(d) suggests that the theory consistently underestimates the magnitude of vortex stretching by turbulence and, hence, the transfer of energy to large wave numbers.

We have found that, to within statistical fluctuations, the evolution of $S(t)$ for a given initial energy spectrum is independent of the initial value of $R_\lambda(0)$ for $20 \leq R_\lambda(0) \leq 45$. This behavior of $S(t)$ is noteworthy as it provides evidence that the small-scale turbulence structure that dominates $S(t)$ is Reynolds-number independent, even at the moderate Reynolds numbers of our numerical simulations. We find that $S \simeq 0.47$ for $1 \leq tv_{\mathrm{rms}}(0)/L(0) \leq 1.75$ when $R_\lambda \geq 20$, in contrast to wind-tunnel experiments that generally give skewnesses in the range 0.3–0.4.[10] Part of the discrepancy is due to the initial peakedness of the spectrum $E(k) \propto k^4 \exp(-Bk^2)$.[5] Higher-resolution numerical experiments are now being performed to examine the skewness at larger Reynolds numbers.

We have also compared time-correlation functions determined by the numerical experiments (Fig. 1) and the direct-interaction theory. There is appreciable disagreement (order 20%) at large wave numbers that apparently can be traced to the deficient transfer indicated by the skewness comparison [Fig. 2(d)]. Low- and moderate-wave-number correlation results are in good agreement. For example, the total correlation at a fixed point in space is given by[5]

$$R(t, t_0) = \frac{\int E(k; t, t_0)\, dk}{\frac{3}{2} v_{\mathrm{rms}}(t) v_{\mathrm{rms}}(t_0)} .$$

For $t_0 = 0.4$, the direct-interaction and run-3C results for $R(t, t_0)$ are essentially identical out to the end of the run at $t = 0.92$ when $R = 0.50$.

Overall, the agreement between the numerical experiments and the direct-interaction approximation, especially for the large-scale dynamics, must be considered an impressive achievement considering the strongly nonlinear nature of the flows.

The resolution of the present simulations is far too low to attempt accurate numerical simulation of inertial-range dynamics[10] such as the Kolmogorov spectrum where $\bar{D}(k) \propto k^{1/3}$. We have found it necessary to retain a long tail in the spectrum of energy dissipation to get accurate solutions of the Navier-Stokes equations; Fig. 2(a) shows that the "inertial range" is confined at present to $2 \ll k \ll 8$! There is evidence[11] that a similar long tail in the spectrum of mean-square vorticity dissipation is necessary for accurate simulations in two space dimensions, so that recent numerical studies[12] of two-dimensional inertial-range dynamics may not be reliable.

This work was begun while the authors were visitors at the National Center for Atmospheric Research, Boulder, Colorado. All the computations were performed at the computer facility there. The comparisons with the direct-interaction approximation were done jointly with Dr. J. R. Herring and Dr. R. H. Kraichnan.

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