Self-Consistent Theory of a Collisionless Resistive Shock

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This paper presents a self-consistent collisionless theory for turbulent, low-Mach-number resistive shocks. Both analytic predictions of shock structure and more detailed numerical solutions are in excellent agreement with experimental observations.

We present a self-consistent collisionless theory of a turbulent resistive shock. The equations for the electron and ion fluids are solved coupled to the ion-acoustic-wave kinetic equations for the turbulent electric fields. We present both approximate solutions determined analytically and more detailed numerical solutions of the structure of a planar steady-state shock. Both analytic and numerical solutions show excellent agreement with experiment. Several novel results from our analyses are summarized as follows: (1) As long as $T_i/T_e \lesssim \frac{1}{2}$, the electron-ion two-stream instability plays a negligible role in determining anomalous resistivity. If $T_i/T_e > \frac{1}{2}$ upstream, the two-stream instability or binary collisions may play a role in preheating the electrons. (2) There is a small amount of resistive ion heating and this plays a crucial role in determining shock structure. (3) The fundamental condition determining the shock width is that ion acoustic waves are at marginal stability throughout the shock. (4) The shock width, and not the resistivity, is the fundamental parameter which we can estimate directly. From the estimate of shock width, the resistivity and turbulent field strength are then obtained. We find that the level of turbulence remains sufficiently low that the linearized wave kinetic equations are valid, and yet the quasilinear resistivity is sufficient to form the shock.

The resistive shock width is given roughly by $L_s \approx \nu_{ei}C^2/\omega_{pe}V_A$ where ν_{ei} is the collision frequency and V_A is the Alfvén speed. Since the classical collision frequency is much too small to explain observed shock widths (typically $10-20c/\omega_{pe}$),[1,2] it has long been recognized that electron-ion streaming instabilities must play an important role.[3-7] Through quasilinear or nonlinear theory, the authors of Refs. 3-7 attempt to find the anomalous resistivity and use it to calculate the shock width. Here we find that the calculation should go in the other direction.

The governing fluid equations (in cgs units) are

$$d(nU)/dx = 0, \tag{1}$$

$$\frac{d}{dx}\left(nMU^2 + nT + \frac{B^2}{8\pi} + \frac{\langle \delta E_y^2\rangle}{8\pi} - \frac{\langle \delta E_x^2\rangle}{8\pi}\right) = 0, \tag{2}$$

$$\frac{d}{dx}\left(\frac{MU^2}{2} + \frac{mV_{ey}^2}{2} + MV_{iy}^2 + 2T + \frac{B_0 B}{4\pi n_0}\right) = 0, \tag{3}$$

$$mU\frac{dV_{ey}}{dx} + MU\frac{dV_{iy}}{dx} - \frac{1}{n}\frac{d}{dx}\frac{\langle \delta E_x \delta E_y\rangle}{4\pi} = 0, \tag{4}$$

$$mU\frac{dV_{ey}}{dx} = \frac{eU}{c}B - \frac{eU_0}{c}B_0 + \frac{e}{n}\langle \delta n_e \delta E_y\rangle, \tag{5}$$

$$(dB/dx) = (4\pi ne/c)(V_{ey} - V_{iy}). \tag{6}$$

We have assumed quasineutrality, so that the x streaming velocity $U(x)$ is the same for electrons and ions. The temperature $T(x)$ is the total electron plus ion temperature: $T(x) = T_e(x) + T_i(x)$. A subscript of zero on any quantity refers to its upstream value. Terms like $\langle \delta E^2\rangle$ correspond to averages taken over fluctuating electric fields in the shock. The turbulent contribution to Eq. (5) enters as a resistivity and plays an important role. The other turbulent contributions are quite small and will be neglected in our analysis, but are retained in the numerical solution of the equations. Finally, to simplify the analysis, we have assumed that heating takes place in two dimensions. The principal effect of this assumption is to lower the critical Mach number $\mathfrak{M}_c$ to about 2.2.[8]

We augment Eqs. (1)-(6) with equations for the turbulent fields. These electric fields are assumed to be generated by ion-acoustic instabilities driven by the dB/dx current. Not only is this the simplest relevant instability, but there is good evidence that the effect of the magnetic field on the instability may be neglected for $\omega_{ce}/\omega_{pe} \ll 1$. The effect of the field is then to prevent stabilization by electron trapping.[7] In this case, the resistive term may be calculated from the quasilinear equation as

$$e\langle \delta E_y \delta n_e\rangle = \sum_{\vec{k}} \frac{2\pi ne^2}{m}\frac{1}{2\pi}\left(\frac{k_y c}{4\pi ne}\frac{dB}{dx} - \omega_L\right)\left(\frac{T_e}{m}\right)^{-3/2}|\varphi(k)|^2, \tag{7}$$

where the summation is only over positive k and $k\varphi(k) = -E(k)$.

In the shock frame, the dispersion relation is given by $\omega = k_x u + \omega_L$, where ω_L is the acoustic frequency in the nondrifting laboratory system,

$$\omega_L = \frac{\omega_{pi}}{\sqrt{2}}\left\{1 + \left[1 + 12(1 + k^2\lambda_{De}^2)\frac{T_i}{T_e}\right]^{1/2}\right\}^{1/2}\frac{k\lambda_{De}}{(1 + k^2\lambda_{De}^2)^{1/2}},$$

valid for $T_e > T_i$.

The wave kinetic equation for each equation for each mode is[9]

$$(d/dx)[V_g N(k, k_0)] = 2\gamma N(k, k_0) + \alpha, \tag{8}$$

where

$$N(k, k_0) = \left(\frac{k^2\omega_{pi}^2}{\omega_L^3} + \frac{6\omega_{pi}^2 k^4 T_i}{M\omega_L^5}\right)\frac{|\varphi(k, k_0)|^2}{2\pi},$$

the action density at x (and wave number k) of a fluctuation which started at $x = -\infty$ with wave number k_0. Since there is no variation in y or t, k_y and ω are constants of the motion, and only k_x changes as the wave propagates. Thus, at any point x where the fluid parameters are $n(x)$, $U(x)$, and $T(x)$, $k_x(x, \omega, k_y)$ may be found in terms of these by inverting the local dispersion relation. The group velocity in the x direction, $V_g(x)$, is simply $U + \partial\omega_L/\partial k_x$.

It never goes to zero in a low-β plasma, since $\partial\omega_L/\partial k_x \sim \sqrt{\beta}\, U$ for ion acoustic waves. The growth rate for these modes is taken to be [10]

$$\gamma = \left(\frac{\pi}{8}\frac{m}{M}\right)^{1/2}\left[\frac{\omega_L}{k(T_e/M)^{1/2}}\right]^3\left[\frac{k_y c}{4\pi n e}\frac{dB}{dx} - \omega_L - \omega_L\left(\frac{M}{m}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2}\exp\left(-\frac{\omega_L M}{2k^2 T_i}\right)\right]\left(1 + \frac{6k^2 T_i}{\omega_L^2 M}\right)^{-1}, \tag{9}$$

valid near marginal stability. It will shortly be clear that the plasma is near marginal stability. The third term in the brackets corresponds to ion Landau damping. Also, α is the rate of thermal excitation, $(\pi m/2M)^{1/2} nT(n\lambda_D^3)^{-1}$.

Physically the evolution of the system is clear. The modes start off at a small thermal level while the fluid attempts to form a soliton.[11] When dB/dx becomes sufficiently large that γ in Eq. (9) becomes positive, the modes grow exponentially until they reach a level large enough to affect the fluid quantities. The principal effect of the waves is to reduce V_{ey} through the last term in Eq. (5), so dB/dx will be reduced and the growth rate shrinks. Hence, a dynamic balance is set up between waves and the fluid. Clearly the nature of this interaction is to drive the waves toward marginal stability. We will now show that marginal stability is maintained locally throughout the shock. To do so, we compare the growth length L_g with the shock width L_s.

These lengths can be compared directly since $dB/dx \sim \Delta B/L_s$. Assuming that $L_g = V_g/\gamma \sim \mathfrak{M} V_A/\gamma$ and that $k \sim k_D$, and neglecting the second term in brackets (ω_L) in Eq. (9), we find that $L_g \sim \mathfrak{M}/c(T_e/m)^{1/2} L_s$. Thus the growth length is much less than the shock width for a nonrelativistic plasma. This means that the equilibrium between waves and fluid will be maintained over a very small scale length compared with the shock width, and the plasma should be at marginal stability locally throughout the shock. Therefore in the shock region, we may use instead of Eq. (5) the marginal stability criterion valid for $T_e/T_i \gtrsim 2.5$,

$$\frac{c}{4\pi n e}\frac{dB}{dx} = \left(\frac{T_e}{2M}\right)^{1/2}\left[1 + \left(1 + 12\frac{T_i}{T_e}\right)^{1/2}\right]^{1/2}\left(1 + \left(\frac{M}{m}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2}\exp\left\{-\frac{T_e}{4T_i}\left[1 + \left(1 + 12\frac{T_i}{T_e}\right)^{1/2}\right]\right\}\right) \tag{10}$$

for purposes of analysis.

If the waves are nearly in a steady state, it can be shown by energy and momentum conservation that the rate of resistive electron heating is $V_{ey}(T_e + 3T_i)^{-1/2}/M$ times the rate of resistive ion heating. However, both electrons and ions heat adiabatically at the same rate ($uT =$ const in a two-dimensional situation). Combining the rates of adiabatic and resistive heating, we obtain an equation relating ion temperature to total temperature:

$$\frac{dT_i}{dx} = \frac{R}{1 + R}\frac{dT}{dx} + \frac{R(T/u)du/dx}{1 + R} - \frac{T_i}{u}\frac{du}{dx}, \tag{11}$$

where

$$R \equiv \frac{V_s}{V_D} = 4\pi ne\left(\frac{T_e}{2M}\right)^{1/2}\left[1+\left(1+12\frac{T_i}{T_e}\right)^{1/2}\right]^{1/2}\left(c\frac{dB}{dx}\right)^{-1}.$$

A simplified description of the shock results from using Eq. (10) in the shock region instead of Eqs. (5) and (6) and the equations for the modes. We do not pursue this analytic treatment in detail here, but use Eq. (10) to get an approximate expression for shock width,

$$L_s \approx \frac{c}{\omega_{pe}}\left(\frac{M}{m}\right)^{1/2}(4\pi nT)^{-1/2}\left(1+\left(\frac{M}{m}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2}\exp\left\{-\frac{T_e}{4T_i}\left[1+\left(1+12\frac{T_e}{T_i}\right)^{1/2}\right]\right\}\right)^{-1}\Delta B. \tag{12}$$

Taking the average of upstream and downstream values of n and T, the remaining parameter to be found is T_e/T_i. If the downstream temperature of the plasma is large compared to the upstream temperature (as occurs when $\mathfrak{M}$ differs appreciably from unity), the temperature ratio should be roughly equal to the ratio of heating rates. Setting the temperature ratio equal to the ratio of resistive heating rates and solving, we find $T_e/T_i \approx 7.4$ for a hydrogen plasma. Thus, Eq. (12) predicts a shock width $L_s \approx 11c/\omega_{pe}$ for upstream parameters $\omega_{pe}/\omega_{ce} = 70$, $\beta = 0.01$, $\mathfrak{M} = 2$. This is in agreement with experiment.[1,2]

Finally, using the estimated value of L_s, the relation between L_s and ν_{ei}, and Eq. (7), one finds an effective resistivity and thereby a value of $\langle(e\,\delta\varphi/T)^2\rangle$. Assuming $k_y \sim \frac{1}{2}k_D$, $\langle(e\,\delta\varphi/T)^2\rangle \approx 0.004$, which also agrees well with experiments.

We now discuss our numerical solutions of Eqs. (1)–(6), (8), and (11). Given $B(x)$ and $dB(x)/dx$, for which we have differential equations, Eqs. (1), (2) and (3) can be solved algebraically to give $n(x)$, $U(x)$, and $T(x)$. Here an iteration is involved, however, since the fluctuation terms, entering through $\langle \delta E_x^2\rangle$, $\langle \delta E_y^2\rangle$, and the drift velocities V_{ey} and V_{iy}, contain dependences on $U(x)$ and $T(x)$.

The fluctuations were determined by integrating Eq. (8) for the wave action density of a large number N_m of modes. Simultaneously Eqs. (5) and (6) were integrated. There were, therefore, $N_m + 2$ coupled first-order ordinary differential equations to be integrated in space from the upstream state through the shock to the downstream state. The integrations were performed in double precision on an IBM 360/91 using a deferred-limit integrator routine[12] with an adjustable convergence parameter. Accuracy of five to six decimal digits was assured by repeating several calculations with different values of this parameter. The number of modes N_m was taken as 16 for most runs, with $N_m = 64$ in a few cases to test the effect of more densely populating k space. Modes were distributed uniformly in angle initially with $|k|$ uniformly spaced to embrace the band of most unstable modes throughout the shock. The soliton solution was reproduced in the absence of the resistivity.

The parameters for the run of Fig. 1 were $\mathfrak{M} = 2.0$, $\beta = 0.01$, $(m/M)^{1/2} = \frac{1}{40}$, $\omega_{pe}/\omega_{ce} = 70$, $T_{eo}/T_{io} = \frac{7}{3}$, and $n\lambda_D^3 = 10^5$ which is artificially high to emphasize the collisionless nature of the solutions. These parameters correspond roughly to the values of the Paul-Daughney-Holmes experiment.[1,2] The shock width seen in Fig. 1, $9c/\omega_{pe}$, agrees closely with the estimate of Eq. (12) and with experiment.

To demonstrate the important role of resistive ion heating, the computation was redone assuming only adiabatic ion heating. Equation (12) now predicts a shock width of roughly $85c/\omega_{pe}$, which is much larger than the experimental values.

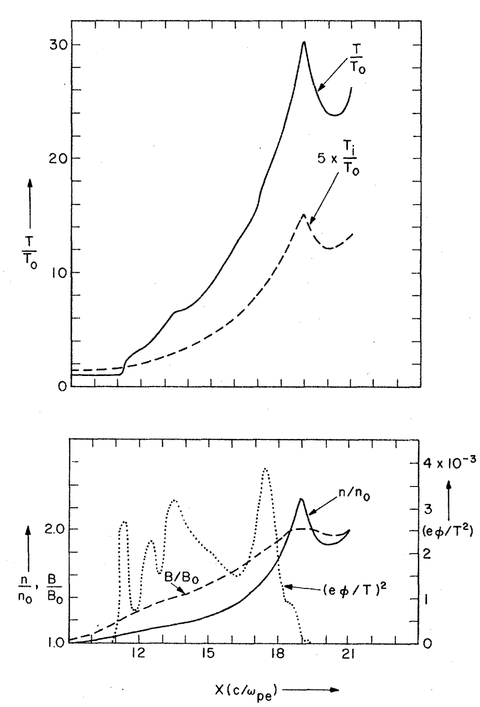

FIG. 1. Structure of a Mach-2 collisionless ion-acoustic shock including resistive ion heating.

The computed width is $75c/\omega_{pe}$. Thus the small amount of resistive ion heating plays a crucial role in determining shock structure.

For the computation shown in Fig. 1, the Rankine-Huginiot conditions are satisfied with small fluctuations about the downstream state, and the preheating of electrons in the early phase of the shock is clearly demonstrated. The value of T_e/T_i averages about 9.5, nearly the value predicted. Furthermore, at no point in the shock is $V_{ey} > (T/m)^{1/2}$, so the electron-ion two-stream instability plays no role anywhere in the shock unless upstream, $T_i \gtrsim T_e/2.5$. Finally, notice that the value of $\langle (e\,\delta\varphi/T)^2\rangle$ is quite close to the predicted value of 0.004.

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High-Order Wave Mixing in Beam-Plasma Instabilities*

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When two waves, ω_1 and $\omega_1+\Delta\omega$, are externally driven in an unstable beam-plasma system, they are found to couple in the nonlinear regime to produce up to ten waves, with frequencies given by $\omega_1 \pm n\Delta\omega$ and amplitudes comparable to the driver amplitudes. This situation is described in terms of third- and higher-order wave-interaction processes. It is suggested that this phenomenon is responsible for nonlinear broadening of the amplified noise spectrum.

We report on the observation of large-amplitude high-order wave interactions between high-frequency electron waves ($\omega > \omega_{ce}$) in a beam-plasma system. When two unstable waves with frequencies ω_1 and ω_2 are driven in the system, satellite waves with frequencies $\omega_n = \omega_{1,2} \pm n\Delta\omega$ are observed, where $\Delta\omega = \omega_1 - \omega_2$ and n is an integer. These satellite waves themselves are (or are close to) eigenmodes of the system, driven by the primary waves via higher-order processes. It is suggested that in the absence of driven waves the same process is responsible for the nonlinear broadening of the noise spectrum.

High-order satellites were also observed by Sato[1] in ion acoustic waves and by Chang, Raether, and Tanaka in beam-plasma waves, as a result of remixing of resonant second-order processes. In the present case the second-order process is nonresonant, but high-order satellite production is resonant and is an important process in the nonlinear regime.

The experiments were done in a hot-cathode dc discharge of diameter 3 cm in a uniform magnetic field in He at 5×10^{-4} Torr. A small electron gun producing a 500-eV beam of diameter 6 mm was placed in the plasma on axis and aligned with the magnetic field. The interaction region has a useful length of 50 cm, where axial density variation is less than 5%. Typical experimental parameters are as follows: electron cyclotron frequency $f_{ce} = 0.5$ GHz, electron plasma frequency $f_p = 0.8$ GHz, and normalized beam density $n_b/n_p = \omega_b^2/\omega_p^2 = 1\times10^{-2}$. Under these conditions