Turbulent Diffusion in Phase Space*

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We formulate a new theory of plasma turbulence which begins by viewing particle diffusion in phase space as a Weiner process. We construct the turbulent particle propagator from physical arguments and find a general expression for the turbulent diffusion coefficient and dielectric function. Earlier results of Dupree and others are extended and clarified. The method readily applies to turbulent problems with boundary conditions on the particle distribution.

It is well known that weak plasma turbulence leads to diffusionlike forms of the kinetic equation. Recent advances in turbulence theory[1,2] have been based on the inclusion of diffusive effects on particle trajectories in phase space. Starting from the Vlasov equation, one obtains coupled equations for the average and stochastic parts of the distribution function. The equation for the stochastic part may be solved by a Green's function, giving an infinite series. In quasilinear theory, the Green's function is the unperturbed particle propagator, $\delta(\vec{R}-\vec{R}_0-\vec{v}_0t)\delta(\vec{v}-\vec{v}_0)$. Unfortunately, because of the singular nature of the propagator, this series does not converge. In the new theory, certain terms in the expansion are included in the definition of the Green's function to account for turbulent effects on the trajectory, giving a convergent perturbation series. The resultant definition for the particle propagator is[2,3]

$$\left[\frac{\partial}{\partial t}+\vec{v}\cdot\frac{\partial}{\partial\vec{R}}-\frac{\partial}{\partial\vec{v}}\cdot\overleftrightarrow{D}(v)\cdot\frac{\partial}{\partial\vec{v}}\right]g\left(\vec{R},\vec{v},\frac{t}{\vec{R}_0},\vec{v}_0,t_0\right)=0,$$
$$g(\vec{R},\vec{v},t_0/R_0,v_0,t_0)=\delta(R-R_0)\delta(v-v_0); \tag{1}$$

$$\overleftrightarrow{D}(v)=(q^2/m^2)\langle E(\vec{R},t)\int_{t_0}^{t}dt\int d\vec{R}'\,d\vec{v}'\,g(\vec{R},\vec{v},t/\vec{R}',\vec{v}',t')E(\vec{R}',t)\rangle. \tag{2}$$

The principal assumptions made in deriving the above equations were the following: (a) The autocorrelation time of the stochastic field is much shorter than the characteristic diffusive spreading time of the propagator. This is a Markovian assumption on the stochastic interaction.[3] (b) Particles are not trapped. Quantitatively, we need $w_b^2/\gamma^2\ll 1$, where w_b is the bounce frequency and γ is the inverse of the correlation time. The latter restriction we take as the definition of weak turbulence. Note that it relates the field amplitude to its width, so that if the spectrum is sufficiently broad, the condition may be satisfied even for large fields.

We wish to show how it is possible, from purely statistical arguments, to derive the form of the particle propagator, the diffusion coefficient, and the plasma dispersion relation. To show the general applicability of this method we then consider two boundary-value problems in phase space.

Consider a test particle in a stochastic field with condition (a) holding. We will find the joint probability distribution function for $(\vec{R},\vec{v},t)$, given $(\vec{R}_0,\vec{v}_0,t_0)$, in one dimension. The point is that this joint probability distribution function is exactly the Green's function that we seek.

Define $v'=v-v_0$ and $R'=R-R_0-v_0\tau$ with $\tau=t-t_0$. Then the equation of motion is

$$\partial v'/\partial\tau=(q/m)\int_0^{\tau}E'(\tau)\,d\tau.$$

We have the immediate result from classical diffusion theory[4] that v' undergoes a diffusion (Weiner) process with the parameter

$$D(v)=(q^2/m^2)\int_0^{\infty}\langle E'(\tau)E'(t+\tau)\rangle\,d\tau, \tag{3}$$

where the integration is to be carried out along a trajectory. Note the equivalence of Eqs. (2) and (3). We assume $D(v)$ is not strongly dependent on velocity, as in earlier work.[1,2] This is consistent with a broad-band turbulent spectrum.

From the above considerations, we can immediately write down the probability distribution function for v':

$$f_{v'}(\tau)=\exp(-v'^2/4D\tau)(4\pi D\tau)^{-1/2}. \tag{4}$$

Since $R' = \int_0^\tau v'(\tau')\,d\tau'$, R' follows an integrated Weiner process, and its variance is given by

$$\langle R'(\tau)R'(\tau)\rangle = \tfrac{2}{3}D\tau^3. \tag{5}$$

This is all we shall need to find the diffusion coefficient.

Next we calculate the joint probability distribution function for (R', v') by noting that the quantity $Z = R' - \frac{1}{2}v'\tau$ is normally distributed and has zero correlation with v', so that

$$f_{R'v'}(\tau) = f_Z(\tau) f_{v'}(\tau) J(Z, v'/R', v'), \tag{6}$$

where J is the Jacobian of the transformation and is 1. Since $\langle Z\rangle = 0$, and by direct calculation $\langle Z(\tau)^2\rangle = \frac{1}{6}D\tau^3$, Eq. (6) becomes[5]

$$g(R', v', \tau) = f_{R'v'}(\tau) = (4\pi D\tau)^{-1/2}\exp(-v'^2/4D\tau)(\tfrac{1}{3}\pi D\tau^3)^{-1/2}\exp[-(R' - \tfrac{1}{2}v'\tau)^2/\tfrac{1}{3}D\tau^3]. \tag{7}$$

The results of Dupree[1] and of Rudakov and Tsytovich[2] are approximations to Eq. (7). They did not allow the initial δ function in velocity to spread. They were able to calculate correct expressions for D because it does not depend on the exact form of the particle propagator, but only on the variance of R'.

Using Eqs. (7) and (2), $D(v)$ may be found; however, we shall derive a general relation from statistical arguments. The integral in Eq. (2) is an expectation value over statistical trajectories so that

$$D(v) = (q^2/m^2)\textstyle\sum_k \langle E_k{}^*E_k\rangle \int_0^\infty d\tau\, e^{i\omega\tau}\langle e^{-ik[R(\tau)-R]}\rangle. \tag{8}$$

We can evaluate the expectation value of the exponential using cumulants:

$$\langle e^{-ik[R(\tau)-R]}\rangle = \exp\left[-ikv - \tfrac{1}{2}k^2\langle R'^2(\tau)\rangle + \sum_{n=3}^{\infty}\frac{C_n}{n!}\langle\langle (ikR')^n\rangle\rangle\right], \tag{9}$$

where C_n is the nth cumulant. Weinstock[6] obtained an expression like Eq. (9) and appealed to weak fields to neglect the cumulants of order 3 and higher. However, if the random-phase approximation is applied, these cumulants are identically zero, as pointed out by Dum and Dupree.[7] Use of the random-phase approximation is equivalent to assuming a normal probability distribution function for the variable, which is certainly correct in this analysis, and in fact in any situation in which is certainly correct in this analysis, and in fact in any situation in which the Markovian assumption holds. Thus a very general expression for the diffusion coefficient is

$$\overleftrightarrow{\mathrm{D}}(v) = (q^2/m^2)\textstyle\sum_k\langle \vec{\mathrm{E}}_k{}^*\vec{\mathrm{E}}_k\rangle \int_0^\infty d\tau\,\exp[i(\omega - \vec{\mathrm{k}}\cdot\vec{\mathrm{v}})\tau - \tfrac{1}{2}\vec{\mathrm{k}}\cdot\langle\vec{\mathrm{R}}'\vec{\mathrm{R}}'\rangle\cdot\vec{\mathrm{k}}]. \tag{10}$$

Now we use Eq. (5) to obtain Dupree's classic result

$$D(v) = (q^2/m^2)\textstyle\sum_k\langle E_k{}^*E_k\rangle\int_0^\infty d\tau\,\exp[i(\omega - kv)\tau - \tfrac{1}{3}k^2D\tau^3]. \tag{11}$$

A very interesting result may be obtained by similar arguments on the dispersion relation. If we use the particle propagator of Eq. (7) in Poisson's equation, the linear dielectric function follows:

$$\epsilon(k,\omega) = 1 - (\omega_p{}^2/ik)\int dv\int d\tau\int dR'\,dv'\,e^{i\omega\tau}g(R', v', \tau)e^{ik[R(\tau)-R]}\,\partial f(v - v')/\partial v. \tag{12}$$

If the diffusive spread in velocity is much less than the thermal width of the distribution function ($D\tau/v_t{}^2 \ll 1$), we can write

$$\frac{\partial f}{\partial v}(v - v') \cong \frac{\partial}{\partial v}\left[f(v) - v'\frac{\partial f}{\partial v}\right] \cong \exp\left(-v'\frac{\partial \ln f}{\partial r}\right)\frac{\partial f(v)}{\partial v}.$$

We expand the expectation value over statistical trajectories as above, and neglect the term $\langle v'^2\rangle(\partial\ln f/\partial v)^2$, since this is of the order $D\tau/v_t{}^2$. Next we interchange the order of integration of τ and v. The integration over v can be changed to an expectation value over particle trajectories by an integration by parts. We have, so far,

$$\epsilon(k,\omega) = 1 - (\omega_p{}^2/ik)\int_0^\infty ik\tau\,d\tau\,\exp(i\omega\tau - \tfrac{1}{2}k^2\langle R'^2\rangle)\langle\langle\exp(-ikv\tau + i\tfrac{1}{2}k\tau\langle v'^2\rangle\partial\ln f/\partial v)\rangle\rangle, \tag{13}$$

where the double angular brackets refer to an expectation value over particle velocities. We expand in cumulants again. For a Maxwellian distribution, the third and higher-order cumulants are zero. For other distribution functions they may be nonzero, but we neglect them here. Then, defining $\langle\langle v^2\rangle\rangle$

$=v_t^2$ and finding that $\langle\langle v\partial \ln f/\partial v\rangle\rangle = -1$, the final result for the dielectric function is

$$\epsilon(k,\omega) = 1 - \omega_p^2 \int \tau\, d\tau \exp(i\omega\tau - \tfrac{1}{2}k^2\tau^2 v_t^2 - \tfrac{1}{2}k^2\langle R'^2\rangle - \tfrac{1}{2}k^2\tau^2\langle v'_2\rangle). \tag{14}$$

This unfamiliar expression becomes the ordinary plasma dielectric function when the correlations go to zero. The integral, in this limit, is an alternate representation of the plasma dispersion function. Equation (14) exhibits clearly the stabilizing effect of turbulent diffusion.[8]

We emphasize the general utility of the above method. To analyze a turbulence problem, one may consider a test particle undergoing a stochastic process in phase space. If the probability distribution function of the velocity can be found, the variance of v' and R' follows, and, using Eqs. (10) and (14), the diffusion coefficient and dispersion relation are obtained in a simple manner.

To demonstrate applicability, we will consider two problems. The first is diffusion with absorbing barriers in velocity space—as may occur, for example, in a mirror-confinement problem. We assume that a particle disappears if its speed is greater than $\pm u$. We can easily obtain the solution for the probability distribution function by the method of images, placing the fundamental solution, Eq. (4), at the initial particle velocity v_0, and at other points outside $(-u, +u)$ to satisfy the boundary conditions. The result is

$$f_v(t) = \sum_{n=0}^{\infty} \frac{\exp\{-[v-(-1)^n v_0 - 2nu]^2/4D\tau\}}{(4\pi D\tau)^{1/2}}, \quad |v|<u; \qquad = 0, \quad |v|>u. \tag{15}$$

After some manipulations, we find the velocity spread

$$\langle v'(\tau)v'(\tau)\rangle = \sum_{n=-\infty}^{\infty} \tfrac{1}{2}\sigma^2 I\left(\frac{u-v_0}{\sigma}, \frac{u+v_0}{\sigma}, \frac{[1+(-1)^n]v_0 + 2nu}{\sigma}\right), \tag{16}$$

where

$$I(a,b,\alpha) = (\tfrac{1}{2}+\alpha^2)[\operatorname{erf}(a-\alpha)+\operatorname{erf}(b+\alpha)] - (a+\alpha)e^{-(a-\alpha)^2}/\sqrt{\pi} - (b-\alpha)e^{-(b+\alpha)^2}/\sqrt{\pi} \tag{17}$$

and $\sigma^2 = 4D\tau$. If we assume $u \gg v_0$ and $u/\sigma \gg 1$, and use the asymptotic expression for $\operatorname{erf}(x)$, then

$$\langle v'(\tau)\rangle = 2D\tau[1 - u(\pi D\tau)^{-1/2}e^{-u^2/4D\tau}], \quad \langle R'(\tau)^2\rangle = \tfrac{1}{3}D\tau^3[1 - 48(\sqrt{D\tau}/u)^3 e^{-u^2/4D\tau}]. \tag{18}$$

The effect of the absorbing barriers is to decrease the correlation functions, and thus increase the diffusion coefficient and give a destabilizing term to the dispersion relation.

Another boundary-value problem of interest in a Q machine is that of a reflecting barrier at $R=0$ and an absorbing barrier at $R=L$. The confining magnetic field is assumed strong enough to allow a one-dimensional model. The boundary conditions are

$$(\partial/\partial R) g^{(R',v',t)}|_{R=0} = 0, \quad g(L-R_0, v', \tau) = 0.$$

The fundamental solution is Eq. (7). The boundary condition at $R=0$ is satisfied by reflecting the solution of the positive side. The boundary condition at $R=L$ is obtained by the method of images. The variance of v' is $2D\tau$, and in the limit that $L - R_0 \gg (D\tau^3)^{1/2}$, we may again use the asymptotic limit for the error function, giving

$$\langle R'^2(\tau)\rangle = \tfrac{2}{3}D\tau^3\left[1 - \frac{(L-R_0)}{8(\frac{1}{3}D\tau^3)^{1/2}} \exp\left(-\frac{(L-R_0)^2}{\frac{4}{3}D\tau^3}\right)\right]. \tag{19}$$

Note that the major effect is that of the absorbing barrier at L, decreasing the R' correlation function, thus increasing D and again adding a destabilizing term to the dispersion relation. Absorbing barriers in R space do not affect $\langle v'(\tau)^2\rangle$, but absorbing barriers in general increase D.

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[8]In previous work (Refs. 1 and 2), for mathematical simplicity the initial $\delta(v - v_0)$ in Eq. (1) is not allowed to spread. The approximation does not affect $D(v)$, since it only depends on $\langle R'^2 \rangle$, but it does affect $\epsilon(k,\omega)$. Dupree obtains Eq. (14) without the $\langle v'^2 \rangle$ term, as a consequence. Thus his diffusive term is $-\frac{1}{3}k^3 D\tau^3$. Substituting $\langle v'^2 \rangle = 2D\tau$ into Eq. (14), we obtain

$$\epsilon(k,\omega) = 1 - \omega_p{}^2 \int \tau d\tau \exp(i\omega\tau - \tfrac{1}{2}R^2 v_t{}^2 - \tfrac{1}{3}k^2 D\tau^3 - k^2 D\tau^3),$$

so our diffusive term $-\frac{4}{3}k^2 D\tau^3$ is 4 times as large as Dupree's.

Electric and Magnetic Field Investigations of the Periodic Gridlike Deformation of a Cholesteric Liquid Crystal

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We have observed a two-dimensional, gridlike deformation of a cholesteric liquid crystal in a magnetic field, proving that this periodic structure does not require space charge or fluid flow. The frequency response of the threshold voltage in an electric field shows that there is a continuous transformation from the pure dielectric to the electrohydrodynamic regime. Investigations near Grandjean-Cano disclinations show the considerable effect of torsional strain on the threshold values.

Several kinds of electric- and magnetic-field-induced distortions are known to occur in cholesteric liquid crystals. A transformation from the cholesteric to the nematic structure was first observed by Wysocki, Adams, and Haas[1] in electric fields and by Sackmann, Meiboom, and Snyder[2] in magnetic fields. The helical unwinding predicted by de Gennes[3] and Meyer[4] for fields perpendicular to the cholesteric helical axis has been experimentally verified for both electric[5,6] and magnetic[7,8] fields. No distortion is expected when fields are applied parallel to the helix axis and the susceptibility anisotropy is negative. Either a 90° rotation of the helical axis or a conical deformation with pitch contraction, depending on the boundary forces, would be expected when the susceptibility anisotropy is positive.[4] The susceptibility anisotropy is defined to be $\Delta\chi = \chi_\parallel - \chi_\perp$, where $\chi_\parallel$ and $\chi_\perp$ are the diamagnetic or dielectric susceptibilities parallel and perpendicular to the local optic axis, respectively. The 90° rotation, making the helical axis perpendicular to the field, happens before helical unwinding and the nematic transformation. This 90° rotation has been observed in electric fields[6,9] and magnetic fields,[10] and the conical deformation has been observed in electric fields.[5] Helfrich[11,12] has discussed the possibility of a periodic, one-dimensional deformation that can occur at even lower fields than the 90° rotation. He theorizes that such deformations could be produced by an electrohydrodynamic process or a purely dielectric process. In the former case, the conductivity anisotropy $\sigma_\parallel - \sigma_\perp$ must be positive, but the dielectric anisotropy $\epsilon_\parallel - \epsilon_\perp$ can have either sign. In the latter process the dielectric anisotropy must always be positive. The purely dielectric case should have a magnetic analog. A two-dimensional periodic deformation has been observed by Gerritsma and Van Zanten[13] and by Rondelez and Arnould.[14] In the latter study the dielectric anisotropy is negative, clearly indicating the electrohydrodynamic process. The dielectric process is probably functioning in the former case.

In this Letter we report the observation of a two-dimensional periodic pattern in a magnetic field (Fig. 1), proving that material flow or electrical conduction plays no role in the deformation. The threshold field data presented help verify Helfrich's predictions for the dielectric process.