Inclusive Approach to Unitarity*

G. Veneziano†
Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 30 December 1971)

We propose a linear sum rule for inclusive-type discontinuities and prove its equivalence to the basic discontinuity equation recently derived by Cahill and Stapp. This, in turn, is believed to be equivalent to unitarity. The possibility of formulating an "inclusive bootstrap" is pointed out.

A basic discontinuity equation has been recently derived from axiomatic (Lehmann-Symanzik-Zimmerman–type) field theory by Cahill and Stapp.[1] In particular cases, this equation yields the much exploited, Mueller-type connection between inclusive cross sections and forward multiparticle amplitudes.[2] In order to describe that equation, consider an arbitrary process $i\to f$ and define the fully connected part of $\langle f|S|i\rangle$ by[3]

$$\langle f|S|i\rangle_c = i(2\pi)^4\delta^{(4)}(P_i-P_f)T(i\to f). \tag{1}$$

We denote by ν_k the channels of the process $i\to f$. A channel is identified by a subset α' of i and a subset α'' of f. Writing $i=\alpha'+\beta'$, $f=\alpha''+\beta''$, our process is written as $\alpha'+\beta'\to\alpha''+\beta''$, where α', α'', β', β'' are arbitrary (possibly empty) sets of multiparticle states. Consider the case in which the vector $Q=P_{\alpha'}-P_{\alpha''}=P_{\beta''}-P_{\beta'}$ is timelike ($Q^2>0$) and define α', α'' so that $Q_0>0$ (energy flows from the set $\alpha\equiv\alpha'+\alpha''$ to the set $\beta\equiv\beta'+\beta''$). Besides the channel $\alpha\to\beta$ (denoted by ν_{Q^2}) the following other channels are present in $T(i\to f)$: (i) The channels ν_α (ν_β) defined by a proper subset of α (β); (ii) the channels ν_γ made up of a proper subset of α and a proper subset of β.

The notation $T(\alpha'+\beta'\to\alpha''+\beta'';\sigma_\alpha,\sigma_\beta,\sigma_\gamma,\sigma_{Q^2})$, where each of $\sigma_\alpha,\sigma_\beta,\sigma_\gamma,\sigma_{Q^2}$ is a set of signs, will define the analytic continuation of T from the physical region (all $\sigma=+1$) to a particular unphysical boundary $[E_{\nu_K}=\mathrm{Re}E_{\nu_K}(1+i\sigma_K\epsilon)]$.

The basic discontinuity equation of Ref. 1 reads

$$T(\alpha'+\beta'\to\alpha''+\beta'';\sigma_\alpha,\sigma_\beta,\sigma_\gamma,\pm)=i(2\pi)^4\textstyle\sum_I T(\alpha'\to\alpha''+I;\sigma_{\alpha+I}{}')\,T(I+\beta'\to\beta'';\sigma_{\beta+I}{}')\,\delta^{(4)}(P_{\alpha'}-P_{\alpha''}-P_I). \tag{2}$$

On the left-hand side of (2), $T(\cdots,\pm)$ stands for $T(\cdots,+)-T(\cdots,-)$ and the signs $\sigma_{\alpha,\beta,\gamma}$ are given. On the right-hand side, $\sum_I$ stands for a completeness sum over intermediate states $(1=\sum_I|I\rangle\langle I|)$, and $\sigma_{\alpha+I}{}'$, $\sigma_{\beta+I}{}'$ are given in terms of σ_α and σ_β in the left-hand side by the following rules[1]:

(1) If $x\subset\alpha$ (x is a proper subset of α), $\sigma_x{}'=\sigma_x$; similarly for $x\subset\beta$.

(2) If $x\subseteq I$, $\sigma_x{}'=+1$ (-1) in $T(\alpha'\to\alpha''+I)$ $[T(I+\beta'\to\beta'')]$.

(2′) If $x\subseteq I$, $\sigma_x{}'=+1$ (-1) in $T(I+\beta'\to\beta'')$ $[T(\alpha'\to\alpha''+I)]$.

(3) If $x=y+z$, $y\subset I$, $z\subset\alpha$, then $\sigma_x{}'=\sigma_z$ if $E_x>0$ and $\sigma_x{}'=\sigma_{\alpha-z}$ if $E_x<0$ (of course, if $P_x^2<0$, $\sigma_x{}'$ is irrelevant). If $x=y+z$, $y\subset I$, $z\subset\beta$, then $\sigma_x{}'=\sigma_z$ if $E_x<0$ and $\sigma_x{}'=\sigma_{\beta-z}$ if $E_x>0$. This third rule is often referred to as the "back-up" rule.

Rules (1), (2), and (3) give one of the basic discontinuity equations of Ref. 1. A second one[4] is obtained by replacing rule (2) with rule (2′). Notice that in either case the signs σ_γ are irrelevant (Steinmann relation[2]) in determining the right-hand side of Eq. (2). In the following we shall omit the label σ_γ. For a more detailed discussion see Ref. 1.

We now claim that Eq. (2) is equivalent to the general sum rule [again $Q^2=(P_{\alpha'}-P_{\alpha''})^2>0$]

$$T(\alpha'+\beta'\to\alpha''+\beta'';\sigma_\alpha,\sigma_\beta,\pm)=2i\pi\textstyle\sum_\gamma\delta(Q^2-\mu_\gamma{}^2)T(\alpha'\to\alpha''+\gamma,\sigma_\alpha)T(\gamma+\beta'\to\beta'',\sigma_\beta)$$
$$+\sum_\gamma\int dP_\gamma\,\frac{E_\gamma}{E_{\alpha'}-E_{\alpha''}}\,T(\alpha'+(\beta',\gamma')\to(\alpha'',\gamma'')+\beta'';\tilde\sigma_{\alpha+\gamma''},\tilde\sigma_{\beta+\gamma'},\pm). \tag{3}$$

In Eq. (3), $\sum_\gamma$ means a sum over all species of stable particles,[3] $P_{\gamma'}=P_{\gamma''}=P_\gamma$, $P_\gamma{}^2=\mu_\gamma{}^2$, and $E_\gamma{}^2=+P_\gamma{}^2+\mu^2$. The signs $\tilde\sigma_{\alpha+\gamma}$ and $\tilde\sigma_{\beta+\gamma}$ will be given by the rules (a), (b), and (c) specified below.

First, we notice that, for $\sigma_\alpha=+1$, $\sigma_\beta=-1$, the sum rule (3) with $\tilde\sigma_{\alpha+\gamma}=+1$, $\tilde\sigma_{\beta+\gamma}=-1$ is identical to the one we have recently proposed[5] and shown to be equivalent to Eq. (2) (in that particular case for

$\sigma_\alpha, \sigma_\beta$). The proof of equivalence will not be repeated here. We only recall that the first term on the right-hand side of (3) gives the pole term (which is therefore an assumption for us), whereas the second term generates the whole of Eq. (2) by induction. The reason why we are led to consider Eqs. (2) and (3) beyond the case $\sigma_\alpha = +1$, $\sigma_\beta = -1$ is because otherwise Eq. (2) is not enough to insure unitarity (as is evident from some examples).

On the other hand, there is a strong feeling among theorists that the whole of Eq. (2) with arbitrary signs $\sigma_\alpha, \sigma_\beta$ would actually imply unitarity.[6] This is confirmed by examples. The problem of generalizing Eq. (3) to arbitrary $\sigma_\alpha, \sigma_\beta$ in such a way as to give Eq. (2) is essentially reduced to finding the rules giving $\tilde{\sigma}$ in Eq. (3) in terms of σ, in a way that induces the rules (1), (2), (2′), and (3) stated above. We have shown that this is accomplished by the following rules:

(a) If $x \subset \alpha\ (\beta)$, $\tilde{\sigma}_x = \sigma_x$.
(b) If $x = \alpha\ (\beta)$, $\tilde{\sigma}_x = +1\ (-1)$.
(b′) If $x = \alpha\ (\beta)$, $\tilde{\sigma}_x = -1\ (+1)$.
(c) If $x = \gamma'' + y$, $(y \subset \alpha)$, $\tilde{\sigma}_x = \sigma_y$ if $E_x > 0$, $\tilde{\sigma}_x = \tilde{\sigma}_{\alpha - y}$ if $E_x < 0$. If $x = \gamma' + y$, $(y \subset \beta)$, $\tilde{\sigma}_x = \sigma_y$ if $E_x < 0$, $\tilde{\sigma}_x = \sigma_{\beta - y}$ if $E_x > 0$.

These rules (a), (b), and (c) complete the definition of our Eq. (3). As we said, the proof is identical to that of Ref. 5 as far as "counting" is concerned. The further thing to check is that rules (1), (2), (2′), and (3) give rules (a), (b), (b′), and (c) and vice versa.

For the first case [(1), (2), (3) ⇉ (a), (b), (c)] one replaces $T(\pm)$ in both sides of Eq. (3) by the right-hand side of Eq. (2). One then gets an equation which is schematically of the form

$$\sum_I T(\alpha + I; \sigma_{\alpha+I}')T(\beta + I; \sigma_{\beta+I}') = \sum_I T(\alpha + I; \tilde{\sigma}_{\alpha+I}')T(\beta + I; \tilde{\sigma}_{\beta+I}'). \tag{4}$$

The equation is satisfied provided $\tilde{\sigma}_x' = \sigma_x'$. This is proven case by case. For instance, the case $x \subset \alpha$ gives

$$\tilde{\sigma}_x' \underset{\text{rule (1)}}{=} \tilde{\sigma}_x \underset{\text{rule (a)}}{=} \sigma_x \underset{\text{rule (1)}}{=} \sigma_x'. \tag{5}$$

The case $x = \gamma'' + y$ $(y \subset \alpha)$ gives similarly

$$\tilde{\sigma}_x' \underset{\text{rule (1)}}{=} \tilde{\sigma}_x \underset{\text{rule (c)}}{=} \sigma_y(E_x > 0) \text{ or } \sigma_{\alpha - y}(E_x < 0) \underset{\text{rule (3)}}{=} \sigma_x'. \tag{6}$$

In this way one can check all cases. The reverse proof [(a), (b), (c) → (1), (2), (3)] goes by induction. First we see that Eq. (3) gives Eq. (2) for $0 < Q^2 <$ the two-particle threshold. Then, using Eq. (3) we prove Eq. (2) up to the three-particle threshold by inserting in the right-hand side of (3) the pole term for $T(\alpha + \beta + \gamma \to \alpha + \beta + \gamma)$. In this way, one sees that Eq. (2) is proven by induction with rules (1), (2), and (3) being forced upon by rules (a), (b), and (c). The proof of equivalence is then completed in the form

$$\text{General sum rules for discontinuities [Eq. (3)]} \Longleftrightarrow \text{General discontinuity equation [Eq. (2)]}. \tag{7}$$

In order to complete our claim of having an inclusive formulation of unitarity one should still show that

$$\text{General discontinuity equation} \Longleftrightarrow \text{Unitarity}. \tag{8}$$

This last problem has been with us for several years.[6] We think that a renewed effort in the direction of proving (8) would be very worthwhile.

Even leaving at present the validity of Eq. (8) as a conjecture, we feel entitled to propose a new approach to strong interaction dynamics, which we call "inclusive bootstrap," in which Eq. (3) replaces the standard form of the unitarity condition[7] and, together with other dynamical assumptions (e.g., Regge behavior, duality), is used to determine self-consistently the parameters of the theory.

Of course, Eq. (3) is equivalent to Eq. (2), and enforcing strictly one will be as hard as enforcing strictly the other. However, the way each equation of type (2) is expressed in terms of equations of type (3), and vice versa, is rather complicated. As a consequence, a perturbative approach starting from the simplest discontinuity equations (say two-body unitarity) and going on to the more complicated ones (n-body unitarity) will be very different from an approach starting from the simplest sum rules (say, a four-point function connected to a six-point function) and progressing to the more complicated ones.

Indeed, as we have stressed already,[5] the second approach looks much simpler. The main reason for that comes from the nonlinear character of Eq. (2) as compared with Eq. (3). This nonlinearity implies the sum over intermediate states I in Eq. (2), which means that, at high-enough energy, amplitudes with any number of legs become involved in each discontinuity equation. Therefore, in the framework of Eq. (2), one cannot imagine a perturbative procedure isolating first amplitudes with a small number of legs. The opposite is true for Eq. (3).

This linearization of unitarity is indeed similar to that achieved by the integral equation of Chew, Goldberger, and Low (or Amati, Fubini, and Stanghellini).[8] Actually, Tan[9] has shown an interesting, though preliminary, derivation of those integral equations from approximations made on the exact Eq. (3). In this approach the pole term in (3) plays the role of the inhomogeneous term in the integral equation.

The viability of this "inclusive bootstrap" seems confirmed by several results recently obtained from particular examples of Eq. (3). On the phenomenological side, information on the correlation functions in many particle spectra has been derived[10] as well as strong constraints on the way scaling is approached in single-particle spectra.[11] On the theoretical side, relationships between intercepts of Regge trajectories and coupling constants have come out easily[12] and many more results seem to be on the way.

Finally, we would like to stress that, with the ever increasing energy of today's experiments, inclusive cross sections might well become the only ones amenable to precise measurements (if not conceptually at least practically). In view of this fact it appears to be worthwhile to try to formulate conservation of probability (i.e., unitarity) in a way that only makes appeal to inclusive quantities.

Very helpful correspondence with H. P. Stapp is gratefully acknowledged.

---

*Work supported in part through funds provided by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-3069.

†On leave from the Weizmann Institute of Science, Israel.

[1]K. Cahill and H. P. Stapp, Berkeley Report No. LBL-381, 1971 (to be published).

[2]A. H. Meuller, Phys. Rev. D 2, 2963 (1970); H. P. Stapp, Phys. Rev. D 3, 3177 (1971); C.-I Tan, Brown University Report No. NYO-2262TA-240 (to be published); J. C. Polkinghorne, to be published.

[3]We use the covariant normalization $\langle P'|P\rangle = (2\pi)^3 2E_P \delta^{(3)}(\vec{P}-\vec{P}')$. The invariant volume element is $dP \equiv d^3P\,(2E_P)^{-1} \times (2\pi)^{-3}$.

[4]We do not consider two of the four forms of the discontinuity equation of Ref. 1. For the purpose of giving unitarity, the two equations we consider are likely to be enough (H. P. Stapp, private communication).

[5]G. Veneziano, Phys. Lett. 36B, 397 (1971).

[6]Interesting steps in this direction are those made by H. P. Stapp, Matscience, Madras, Report No. 26, 1964 (unpublished), and International Centre for Theoretical Physics, Trieste, Report No. 1C/65/17, 1965 (unpublished); and by D. I. Olive, Nuovo Cimento 37, 1422 (1965).

[7]Use of inclusive sum rules for dynamical purposes was first discussed by G. Veneziano, Phys. Lett. 34B, 53 (1971).

[8]G. F. Chew, M. L. Goldberger, and F. E. Low, Phys. Rev. Lett. 22, 208 (1969); D. Amati, S. Fubini, and A. Stanghellini, Nuovo Cimento 26, 896 (1962).

[9]C.-I Tan, Brown University Report No. NYO-2262TA-254, 1971 (to be published).

[10]E. Predazzi and G. Veneziano, Lett. Nuovo Cimento 2, 749 (1971); S. H.-H. Tye, Massachusetts Institute of Technology Report No. CTP 239, 1971 (to be published); K. H. Biebl and J. Wolf, Phys. Lett. 37B, 197 (1971); A. Bassetto and M. Toller, CERN Report No. TH-1337, 1971 (to be published).

[11]S. H.-H. Tye and G. Veneziano, Massachusetts Institute of Technology Report No. CTP-243, 1971 (to be published); L. Caneschi, CERN Report No. TH-1416, 1971 (to be published).

[12]C. E. DeTar, D. Z. Freedman, and G. Veneziano, Phys. Rev. D 4, 906 (1971); C. E. Jones, F. E. Low, G. Veneziano, and J. E. Young, to be published.