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Feynman-Graph Expansion for Critical Exponents*

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The critical exponents γ, η and the "crossover index" φ are computed for generalized classical Heisenberg models with n internal degrees of freedom as an exact expansion in $\epsilon = 4 - d$ (d is the number of space dimensions). Results are obtained to order ϵ^2 for γ and to order ϵ^3 for η. The results to this order for the three-dimensional Ising case ($n = \epsilon = 1$) are $\gamma = 1.244$ and $\eta = 0.037$.

In a previous Letter[1] Fisher and the author obtained expansions for critical exponents in powers of $\epsilon = 4 - d$, where d is the dimensionality of the system. Generalized Ising and Baxter models were studied using an approximate renormalization-group recursion formula.[2] The results were exact to order ϵ but in error in order ϵ^2.

In this paper exact expansions of critical exponents are reported to order ϵ^2 at least; they were obtained by Feynman-graph techniques. Generalized classical Heisenberg models are discussed; the spin s has n intermal indices. The conventional Heisenberg model corresponds to $n = 3$; the Ising model corresponds to $n = 1$. The results are as follows:

$$\gamma = 1 + \frac{(n+2)}{2(n+8)}\epsilon + \frac{(n+2)(n^2+22n+52)}{4(n+8)^3}\epsilon^2 + O(\epsilon^3), \tag{1}$$

$$\eta = \frac{(n+2)}{2(n+8)^2}\epsilon^2 + \frac{(n+2)}{2(n+8)^2}\left[\frac{6(3n+14)}{(n+8)^2} - \frac{1}{4}\right]\epsilon^3 + O(\epsilon^4), \tag{2}$$

$$\varphi = 1 + \frac{n\epsilon}{2(n+8)} + \frac{\epsilon^2(n^3+24n^2+68n)}{4(n+8)^3} + O(\epsilon^3), \tag{3}$$

where φ is the "crossover index" of Riedel and Wegner.[3] The numerical results for $\epsilon = 1$ ($d = 3$) from these series are shown in Table I. The value for φ to order ϵ has previously been obtained by Wegner[6] and by Fisher and Pfeuty.[5]

The Feynman-graph method of this paper is unrelated to the renormalization-group methods of Refs. 1 and 2; in particular the calculation of η described here is distinct from the exact renormalization-group calculation mentioned in Ref. 1. However, a renormalization-group argument will be used below to motivate one step in the Feynman-graph calculation. The calculation of critical exponents in powers of ϵ is simpler in the graphical approach than in the exact renormalization-group approach. The renormalization-group approach remains important for other problems, such as determining the domain of initial Hamiltonians associated with a given set of exponents (see Ref. 1).

The method of calculation will be described briefly. The Hamiltonian used was similar to that of Ref. 1; we define

$$H/kT = \int\{\tfrac{1}{2}r_0 s^2(\vec{x}) + \tfrac{1}{2}[\nabla s(\vec{x}) - \nabla\nabla^2 s(\vec{x})]^2 + u_0 s^4(\vec{x})\}d^d x, \tag{4}$$

where $s(\vec{x})$ is a spin field with n components $s_i(\vec{x})$ [s^2 means $\sum_i s_i^2$ and s^4 means $(\sum_i s_i^2)^2$] and r_0 and u_0 are constants. The term $\nabla\nabla^2 s(\vec{x})$ is present to make integrals converge; its effect is to suppress fluc-

TABLE I. Numerical results in three dimensions.

n	Exponent	ϵ series	Best estimate
1	γ	1.244	1.250 ± 0.003 [a]
3	γ	1.347	1.375 ± 0.01 [a]
∞	γ	1.750	2 [b]
1	η	0.037	0.055 ± 0.01 [a]
3	η	0.039	0.043 ± 0.14 [a]
∞	η	0	0 [b]
3	φ	1.22	... [c]
∞	φ	1.75	2 [b,d]

[a] These are high-temperature series estimates taken from Ref. 9, Table 6.III.
[b] These are exact: H. E. Stanley, Phys. Rev. 176, 718 (1968).
[c] See Fisher and Pfeuty (Ref. 5).
[d] See the first paper of Ref. 3.

tuations with wavelengths $\ll 1$.[7] The temperature is absorbed into the constants r_0 and u_0, and r_0 is varied to find the critical point.

Several correlation functions were calculated near the critical point as an expansion in both u_0 and ϵ. How this expansion was used to generate critical exponents in explained later. Thousless[4] and Larkin and Khmel'nitskii[8] have discussed Feynman rules for computing the u_0 expansion. The continuation to nonintegral d was performed diagram by diagram. One has to define integrals of the form $(2\pi)^{-d}\int d^dk \times f(k^2, \vec{k}\cdot\vec{k}_1, \cdots, \vec{k}\cdot\vec{k}_m)$ for various functions f and various numbers of external momenta $\vec{k}_1, \cdots, \vec{k}_m$. The calculations reported here involved only integrals of the form $(2\pi)^{-d}\int d^dk\, f(k^2)$ and $(2\pi)^{-d}\int d^dk\, f(k^2, \vec{k}\cdot\vec{k}_1)$. For any integer d one can write

$$(2\pi)^{-d}\int d^dk\, f(k^2) = K_d \int_0^\infty k^{d-1} f(k^2)\, dk, \tag{5}$$

$$(2\pi)^{-d}\int d^dk\, f(k^2, \vec{k}\cdot\vec{k}_1) = (2\pi)^{-1} K_{d-1} \int_0^\infty dk \int_0^\pi d\theta\, k^{d-1} (\sin\theta)^{d-2} f(k^2, k_1 k\cos\theta), \tag{6}$$

with $K_d = 2^{-(d-1)}\pi^{-d/2}[\Gamma(d/2)]^{-1}$. These formulas were used for nonintegral d. They can be generalized if necessary to the case $m > 1$.

Let enclosure by angular brackets denote the thermodynamic average; the averaging in this case involves a functional integral over all spin functions $s(\vec{x})$ and division as usual by the partition function. Then the correlation function $g(\vec{k})$ is

$$g(\vec{k}) = \int e^{i\vec{k}\cdot\vec{x}} \langle s_i(\vec{x}) s_i(0)\rangle\, d^dx \tag{7}$$

(not summed over i; the answer is independent of i). The other correlation functions used in this paper were the four-spin correlation function $g(\vec{k}, \vec{k}_1, \vec{k}_2)$ [a triple Fourier transform of $\langle s_i(\vec{x}) s_i(\vec{x}_1) s_i(\vec{x}_2) s_i(0)\rangle - \langle s_i(\vec{x}) s_i(\vec{x}_1)\rangle\langle s_i(\vec{x}_2) s_i(0)\rangle$ – (two similar terms)], and a correlation function involving $\langle s_i(\vec{x}) s_j(\vec{x}_1) s_i(0) \times s_j(0)\rangle$ $(j \neq i)$ which was used in calculating φ.

In the perturbation calculation one calculates these correlation functions in powers of u_0. However, it simplifies the calculation if one defines the perturbation to include a term $\int r_1 s^2(\vec{x})\, d^4x$ so that the unperturbed interaction involves a constant $r = r_0 - r_1$ in place of r_0. The constant r is chosen to be proportional to the reciprocal of the susceptibility so r vanishes at the critical point.

The quantities calculated explicitly were $g(\vec{k})$ for $\vec{k} = 0$, $g(\vec{k})$ for $r = 0$ and $k^2 \ll 1$, the correlation function needed to compute φ, and the constant $u_R = -g(0, 0, 0)/24g^4(0)$ which in field-theoretic terminology is (essentially) the renormalized coupling constant. The condition on r is explicitly that $g(0)$ be $1/r$ independent of u_0; the graphs that would have contributed to $g(0)$ are canceled by choosing r_1 properly.

The quantity r_1 is obtained from the graph calculation as a function of r, u_0, and ϵ. Given r_1, one knows r_0, namely, $r_0 = r + r_1(r, u_0, \epsilon)$. The critical value of r_0 is $r_{0c} = r_1(0, u_0, \epsilon)$. To determine the susceptibility index γ, one must calculate how $r_1(r, u_0, \epsilon)$ varies with r. It is convenient to write

$$r_0 - r_{0c} = r_1(r, u_0, \epsilon) - r_1(0, u_0, \epsilon) = rR(r, u_0, \epsilon).$$

For $r_0 \to r_{0c}$ one expects $r \propto (r_0 - r_{0c})^\gamma$. Thus $R(r, u_0, \epsilon)$ should behave as $r^{\gamma^{-1}-1}$ for $r \to 0$.

The result of the graph calculation for u_R is

$$u_R = u_0 + (n+8)\frac{u_0^2}{4\pi^2}\left[\ln r + \tfrac{17}{6} - \tfrac{1}{4}\epsilon \ln^2 r - \epsilon(\tfrac{1}{2} + K')\ln r\right] + \frac{u_0^3}{16\pi^4}\left[(n+8)^2(\ln r + \tfrac{17}{6})^2 - (20n+88)\ln r\right] + \cdots, \quad (8)$$

where K' is defined by $K_{4-\epsilon} = K_4(1 - \epsilon K') + O(\epsilon^2)$. The terms neglected include terms of order u_0^4, $u_0^3\epsilon$, $u_0^2\epsilon^2$, or higher; terms of order u_0^3 and $u_0^2\epsilon$ but independent of r; and all terms of order r or smaller for small r. Similar results were obtained for the other correlation functions, except that in $g(\vec{k})$ with $r = 0$, logarithms of k replace logarithms of r.

In the critical region, $g(\vec{k})$ should behave as $k^{-2+\eta}$. The Widom-Kadanoff scaling laws[9] can be used to predict the behavior of u_R in the critical region; the result is that u_R is proportional to $r^{(\epsilon-2\eta)/(2-\eta)}$. The exponents γ and η can depend on ϵ but should be independent of u_0. All these statements have been verified to all orders in ϵ using renormalization-group arguments.[10] The explicit expansions in u_0 are obviously not independent of u_0 and normally one must sum the complete series in u_0 to see the critical behavior independent of u_0. However, a nontrivial renormalization-group argument (given later) shows that *there is a unique choice* $u_0 = u_0(\epsilon)$ *for* u_0 *for which the explicit expansions in* u_0 *match the expected critical behavior.* Consider, for example, the renormalized coupling constant u_R. The function $u_0(\epsilon)$ is itself a power series in ϵ; when this is substituted in Eq. (8), u_R becomes a power series in ϵ with coefficients depending on $\ln r$: $u_R = u_R(\epsilon, \ln r)$. The theorem is that $u_R(\epsilon, \ln r)$ is proportional to $r^{(\epsilon-2\eta)/(2-\eta)}$ for any r, with a proportionality factor independent of r. This means

$$u_R(\epsilon, \ln r)/u_R(\epsilon, 0) = \exp[(\epsilon - 2\eta)(\ln r)/(2-\eta)]. \quad (9)$$

This formula is required to hold order by order in ϵ (taking into account that η depends on ϵ).

The functions $u_0(\epsilon)$, $\eta(\epsilon)$, and $\gamma(\epsilon)$ are determined by this matching condition plus the corresponding matching conditions that $g(k)$ behave as $k^{\eta-2}$ and $R(r, u_0(\epsilon), \epsilon)$ behave as $r^{\gamma^{-1}-1}$. To order ϵ, for example, one finds $\eta = 0$, so Eq. (9) reduces to

$$1 + (n+8)(u_0/4\pi^2)\ln r = 1 + \tfrac{1}{2}\epsilon \ln r, \quad (10)$$

and hence $u_0(\epsilon) = 2\pi^2\epsilon/(n+8)$ to order ϵ. Equations (1)–(3) result from the solution of matching conditions like this.

Now the existence of $u_0(\epsilon)$ will be explained. Consider the renormalization-group recursion formula of Ref. 1:

$$u_{l+1} = u_l + (\epsilon \ln 2)u_l - 9u_l^2. \quad (11)$$

The constant u_l is the effective (field-theoretic) coupling constant for blocks of spins of size 2^l. In the critical region u_l is a constant independent of l, namely, $u_l = u^* = \frac{1}{9}\epsilon \ln 2$. If one chooses $u_0 = u^*$, then the critical region includes all l (not quite; see below). However, if u_0 is a little different from u^*, say, $u_0 = u^* + B$ with B small, then solution of the linearized recursion formula[1] gives

$$u_l = u^* + B\exp(-\tfrac{1}{9}\epsilon l \ln 2). \quad (12)$$

In this case $u_l \to u^*$ only for $\epsilon l \gg 1$. Blocks of spins of size 2^l correspond to fluctuations with momenta $k \sim 2^{-l}$, so this means the critical region is only the region $|\epsilon \ln k| \gg 1$, and similarly one must have $|\epsilon \ln r| \gg 1$.

When one studies the exact renormalization-group recursion formula (see Ref. 1), it is seen that there are many initial transients which prevent u_l from being constant for all l including $l = 0$, but there is only one term which falls off as slowly as $e^{-\epsilon l}$. Choosing $u_0 = u_0(\epsilon)$ ensures that the coefficient of the slow transient is zero. [The other transients have been removed from the expansion in Eq. (8); other transients correspond to terms going to zero as a power of r.]

To test the convergence of the expansion in ϵ, the expansion for γ has been calculated to eighth order using the approximate renormalization-group recursion formula for the Ising case[2] ($n = 1$). The series is obviously asymptotic rather than convergent: The coefficients of ϵ, ϵ^2, etc. are, respectively, 0.167, 0.041, −0.016, 0.077, −0.2, 0.67, −2.5, 10.3. The exact series is also expected to be asymptotic: One cannot expect a convergent series when expanding an s^4 term relative to an s^2 term inside an integral over all s. (For $n = \infty$, the exact ϵ series is convergent, but this is presumably true only of the $n = \infty$ limit.)

The ϵ expansion should be a powerful tool for studying many aspects of critical phenomena because it can be computed from Feynman graphs. Graphs can be defined for just about any system and any problem no matter how complex.

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[9]For a review see M. E. Fisher and D. Jasnow, "Theory of Correlations in the Critical Region" (to be published).

[10]An exact formulation of the renormalization group is mentioned in Ref. 1.

Acoustoelectric Periodic Structure

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The interaction of a conducting pseudoperiodic grating with volume acoustic modes of a piezoelectric crystal has been tentatively explained in terms of an acoustoelectric periodic structure exhibiting exponential and periodic couplings.

The physical appearance of the interaction between a pseudoperiodic metallic grating, deposited along a pure-mode axis of a piezoelectric crystal, and volume acoustic modes was documented in a recent experimental investigation.[1] The present note suggests that this kind of interaction can be regarded as an acoustoelectric periodic structure[2] exhibiting exponential and periodic couplings.

The sample used in the present study consisted of two 25-pair pseudoperiodic gratings with a spatial period of $2\pi/k_T = 32\times10^{-6}$ m (where k_T is the grating wave number) deposited 1.5×10^{-3} m apart along the Z axis of a Y-cut oriented $LiNbO_3$ crystal. For further details the reader is referred to Ref. 1. The transmission spectrum of the sample is shown in Fig. 1(a). The spectrum can be related to the quasistatic dispersion relation for the acoustic wave propagation along the Z axis of a semi-infinite crystal, which can be written as

$$D(\omega,k_z)\equiv D_R(\omega,k_z)D_s(\omega,k_z)D_l(\omega,k_z)\equiv(\omega^2-v_R{}^2k_z{}^2)(\omega^2-v_s{}^2k_z{}^2)^2(\omega^2-v_l{}^2k_z{}^2)=0, \tag{1}$$

where ω is the angular frequency, k_z is the longitudinal wave number, v_R is the velocity of the Rayleigh surface wave, v_s is the velocity of a degenerate set of pure-shear volume modes, and v_l is the velocity of the volume longitudinal piezoelectric mode. The periodic perturbation introduced in the crystal by conducting gratings results in the periodicity of the dispersion equation which can be written in a coupled form as

$$D^c(\omega,k_z)\equiv D_R{}^c(\omega,k_z+Lk_T)D_s{}^c(\omega,k_z+Mk_T)D_l{}^c(\omega,k_z+Nk_T)=0, \tag{2}$$

where L, M, and N are integers. This periodicity accounts for couplings between each of the acoustic modes as shown in Fig. 1(b), which was obtained by a graphical procedure applicable to periodic structures.[2] In Fig. 1(b) the Brillouin zone along the Z axis is shown for the three acoustic branches of which one is degenerate. The remarkable feature of the coupled dispersion equation is that forbidden bands give rise to enhanced couplings. Each splitting is due to an interaction of two branches of a specific acoustic mode yielding for real ω two complex wave numbers, one of which is responsible for a growing mode. Gaps containing two splittings at the zone edge will generate two growing modes propagating in the opposite directions. The modes will grow exponentially in space along the grating structure (Z direction) on account of the energy supplied to the grating, which is termed exponential coupling. Two more points are of interest here. First, the coupling due to splittings located at the center of the zone, $|k|\approx 0$, are rather small, since the growth parameter $|\mathrm{Im}(k)|d \ll 1$, where d is the length of the grating, while at the edge of the zone, $|k|\simeq k_T$, the couplings are strong