Ion Runaway in Tokamak Discharges*

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The Tokamak discharge, with its characteristic "enhanced resistivity," can permit singly charged ions to maintain themselves in the applied electric field at energies considerably above the mean thermal energy.

The resistivity of the Tokamak discharge[1] in hydrogen is typically 3–10 times higher than one would calculate on a simple classical model.[2] In the standard Tokamak regime the discrepancy factor appears to be due largely to high-Z impurity ions.[3] In addition the "neoclassical" theory of toroidal transport coefficients in the long-mean-free-path regime predicts an enhancement of resistivity[4] due to the magnetic trapping of electrons. An experimental effect roughly consistent with this prediction has been reported.[5] In the present paper we show that enhanced resistivity due to high-Z impurities or magnetic trapping implies the possibility of ion runaway.

We will consider, first of all, a plasma discharge in a uniform magnetic field $\boldsymbol{B}$ and a uniform parallel electric field E. The electric field causes a mean electron drift velocity v_{ed} which will be assumed small compared to the electron thermal velocity v_{eT}. The equation of motion of a test ion with mass M and a charge Z is then given by $M dv/dt = eZE - F_e - F_i$. Here F_e is the total electron friction, which we may separate into a term F_{ed} that is due to the electron drift, plus a term F_{e0} that represents the friction that the test ion would experience if the electrons were at rest. The equilibrium of the electrons requires $n_e eE = n_i F_{ed}(Z_i)$, where n_i and Z_i refer to the *bulk* plasma ions. Since the functional dependence of F_{ed} on ion charge is expressed by $F_{ed}(Z) = Z^2/Z_i^2 F_{ed}(Z_i)$, and since $Z_i n_i = n_e$, we thus have $F_e = e(Z^2/Z_i)E + F_{e0}$. The frame is chosen so that the bulk ions are at rest ($v_{id}=0$, $F_i = F_{i0}$). We have assumed also that $v \ll v_{eT}$. It is convenient to introduce an effective electric field $\boldsymbol{E}^* = E(1 - Z/Z_i)$ and write

$$M dv/dt = eZE^* - F_{e0} - F_{i0}. \qquad (1)$$

If the test ion has the same charge as the bulk ions, then E^* vanishes and the frictional forces always cause the test ion to come to rest. This is the simple classical case. As was pointed out some time ago by Gurevich,[6] a test ion with $Z > Z_i$ experiences an effective electric field E^* in the direction opposite to E, which can be much larger than E and can cause the test ion to run away in the direction of electron streaming. In the context of the Tokamak, we are interested in the complementary case: A test ion with $Z < Z_i$ sees an effective field E^* in the same direction as $\boldsymbol{E}$, and of nearly the same magnitude, if $Z_i \gg Z$. Whether runaway occurs depends of course on the competition of the $\boldsymbol{E}^*$ term and the frictional drag terms in Eq. (1). For small v the ion drag predominates, and for large v the electron drag becomes large. The sum of the two drag terms has a minimum at $v_m = 3^{1/3}(2\pi)^{1/6} Z_i^{1/3} T_e^{1/2} / M_i^{1/3} m_e^{1/6}$. The right-hand side of Eq. (1) can become positive near this minimum, provided we satisfy the condition

$$E^* = E_c(3m_e/2\pi Z_i^2 M_i)^{1/3}, \qquad (2)$$

where $E_c = 4\pi n_i Z Z_i^2 e^3 \ln(\Lambda)/T_e$. A test ion in the velocity range between the two nulls v_1 and v_2 of dv/dt will tend to run away toward v_2. A test ion initially at high energy will tend to slow down and approach v_2 from above. (Test ions with velocity exceeding v_{eT} can, of course, run away entirely, but this is not a case of practical interest for the Tokamak discharge.) In actual Tokamak experiments, the situation is slightly different from the above model: The bulk plasma consists mainly of hydrogen ions, and the "effective Z" factor in the resisitivity is contributed by a small percentage of high-Z ions. The above results still apply, however, if Z_i is identified with the measured effective Z, but M_i is identified with the mass of hydrogen. It is easy to verify numerically that condition (2) is generally satisfied by present-day Tokamak experiments.[1,3]

We now turn to the effect of magnetic trapping. Assuming a simple model where the current is carried by the untrapped electrons, we have for the electron equilibrium condition $n_e^u eE = n_i F_{ed}^i + n_e^t F_{ed}^t$, where n_e^u refers to the untrapped electron density, F_{ed}^i refers to the friction of the ions against the electron streaming, and n_e^t and F_{ed}^t

refer to the density of the trapped electrons and the friction that they contribute. The effective electric field E^* is given by

$$E^* = E\left(1 - \frac{Zn_e{}^u/Z_i n_e}{1 + F_{ed}{}^t n_e{}^t / F_{ed}{}^i n_e}\right),$$

which is to be used in Eq. (1) as before. Note that the nonvanishing of E^* for $Z=Z_i$ is due to two separate effects: the nonequality of $n_e{}^u$ and $Z_i n_i$, and the scattering of drifting electrons on trapped electrons. If the trapping is due to a small magnetic field variation ΔB along magnetic field, we can estimate that $n_e - n_e{}^u = n_e{}^t \sim (\Delta B/B)^{1/2}$, so that for $Z=Z_i$ we have $E^* \sim 2(\Delta B/B)^{1/2}E$. For axisymmetric Tokomaks, neoclassical transport theory in the lowest collision frequency regime has yielded the electron distribution function in the presence of a driving E field[7]; using this distribution to calculate the electron friction on the ions, we obtain $E^* = 1.6(r/R)^{1/2}E$ for Lorentz gas electrons and $E^* = 2.4(r/R)^{1/2}E$ with electron-electron collisions included.

Having established the possibility of ion runaway in Tokamak discharges, we must next consider quantitatively the effect on the ion distribution function. In ordinary thermal equilibrium, ions diffuse into the runaway region mainly from below and are lost from it by scattering of the velocity vector away from the direction of the electric field. This problem was analyzed by Gurevich[6] neglecting the pitch-angle scattering; here, we follow a method previously applied to the electron-runaway problem.[8] Scaling the velocity v to a runaway velocity $v_c = (4\pi n_i Z Z_i^2 e^3 \ln(\Lambda)/M_i E^*)^{1/2}$ by writing $u = v/v_c$ and using $\mu = \cos\theta$, the Fokker-Planck equation for the runaway ion distribution including scattering in pitch angle θ becomes

$$\mu\frac{\partial f}{\partial u} + \frac{1-\mu^2}{u}\frac{\partial f}{\partial \mu} = \frac{\lambda}{2u^3}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu} + \frac{1}{u^2}\frac{\partial}{\partial u}\left[(1+au^3)f + \frac{\epsilon}{u}(1+\tau a u^3)\frac{\partial f}{\partial u}\right], \tag{3}$$

where $\lambda = M_i/M$, $\tau = T_e/T_i$, $a = [\frac{2}{3}(2\pi)^{1/2}Z_i](m/M_i)^{1/2}(E_c/E^*)^{3/2}$, and $\epsilon = (T_i M_i/T_e M)E^*/E_c \ll 1$. We write

$$f(u,\mu) = \exp[-\varphi(u,\mu)]; \quad \varphi(u,\mu) = \varphi_0(u) + (1-\mu)\varphi_1(u) + \cdots. \tag{4}$$

It is clear that for $u \sim 1$ the distribution must be exponentially small in ϵ: The appropriate ordering is in fact $\varphi_0 \sim \epsilon^{-1}$ and $\varphi_1 \sim \epsilon^{-1/2}$; with these assumptions we may substitute Eq. (4) into Eq. (3) and select the dominant terms in ϵ to obtain straightforwardly

$$\begin{aligned}\varphi_0 = \varphi_0{}^{(1)}(u) &= \frac{1}{\epsilon}\int_0^u \left(\frac{1+au^3-u^2}{1+\tau a u^3}\right)u\,du + \frac{\lambda^{1/2}}{\epsilon^{1/2}}\int_0^u \frac{u\,du}{(1+au^3-u^2)^{1/2}(1+\tau a u^3)^{1/2}},\\ \varphi_1 &= u^2(1+au^3-u^2)^{1/2}/\epsilon^{1/2}\lambda^{1/2}(1+\tau a u^3)^{1/2}.\end{aligned} \tag{5}$$

The lower limits of the integrations over u in Eqs. (5) are determined by matching the distribution to a Maxwellian at small u: For $u \sim \epsilon^{1/2}$ we have $\varphi = u^2/2\epsilon + O(\epsilon^{1/2}) = Mv^2/2T_i + O(\epsilon^{1/2})$. The solutions given in Eqs. (5) are valid only if $1 + au^3 - u^2 > 0$; if $a < 2/3\sqrt{3} \simeq 0.4$ [condition (2)], there are two velocities u_1 and u_2 where $1 + au_{1,2}{}^3 - u_{1,2}{}^2 = 0$. The region $u_1 < u < u_2$ is the runaway region. Before considering this runaway region it is necessary to investigate a boundary layer around $u = u_1$: The appropriate ordering for this boundary layer is $u - u_1 \sim \epsilon^{1/3}$, $\varphi_0 \sim \varphi_1 \sim \epsilon^{1/3}$; with these assumptions we obtain

$$1 - (2-3au_1)(u-u_1)\varphi_1 u_1^{-1} - \epsilon\lambda(1+\tau a u_1^3)\varphi_1^3 u_1^{-6} = 0;$$

$$\varphi_0 = -\lambda\varphi_1[1 + \epsilon\lambda\varphi_1^3(1+\tau a u_1^3)/2u_1^6]/u_1^2(2-3au_1) + \text{const}.$$

For large $u_1 - u$ we have

$$\varphi_1 \simeq u_1^{5/2}(2-3au_1)^{1/2}(u-u_1)^{1/2}/\epsilon^{1/2}\lambda^{1/2}(1+\tau a u_1^3)^{1/2};$$

$$\varphi_0 \simeq \frac{-u_1^2(2-3au_1)(u_1-u)^2}{2\epsilon(1+\tau a u_1^3)} - \frac{2\lambda^{1/2}u_1^{1/2}(u_1-u)^{1/2}}{\epsilon^{1/2}(2-3au_1)^{1/2}(1+\tau a u_1^3)^{1/2}} + \varphi_0{}^{(1)}(u_1).$$

These correctly match with the solutions (5) for small $u_1 - u$. The boundary layer solution for large $u - u_1$ which must be matched to the solution in the runaway region is then

$$\varphi_1 \simeq u_1/(2-3au_1)(u-u_1); \quad \varphi_0 \simeq -\lambda/u_1(2-3au_1)^2(u-u_1) + \varphi_0{}^{(1)}(u_1).$$

In the runaway region $u_1 < u < u_2$ the distribution is not a rapidly varying function: Indeed the appropriate ordering is $\varphi_0 \sim 1 \sim \varphi_1 \sim \varphi_2 \cdots$; with this assumption we obtain equations for the exponents φ_n, the first of which is $\lambda\varphi_1 = -u(1+au^3-u^2)\partial\varphi_0/\partial u + 3au^3$. These do not, however, form a closed set. Integrating Eq. (2) to lowest order over μ and u we obtain the constraint $u^2\int_{-1}^{1}\mu f\,d\mu = (1+au^3)\int_{-1}^{1}f\,d\mu + \text{const}$, and the integration constant (the total flux) must vanish in steady state. If we suppose that the distribution in the runaway region is concentrated around $\mu=1$, then this constraint yields a solution for φ_1 from which φ_0 may then be obtained:

$$\varphi_1 = -u^2/(1+au^3-u^2)\,; \quad \varphi_0 = \int^u u\,du[\lambda + 3au(1+au^3-u^2)]/(1+au^3-u^2)^2.$$

For small $u-u_1$ these solutions correctly match to the boundary layer solutions already given. At the upper limit of the runaway region, i.e., around $u=u_2$, a secondary layer occurs which may be treated exactly like the boundary layer around $u=u_1$. Finally, for $u>u_2$ where $1+au^3-u^2>0$ again, the solutions are exactly like those given in Eqs. (5) except that the lower limits of integration will be u_2, and the constant $\varphi_0^{(1)}(u_1)$ must be added to φ_0; of course the distribution rapidly vanishes in this outer region.

In the extreme case where $a \ll 0.4$ we have $u_1 \simeq 1$ and $u_2 \simeq a^{-1}$; the magnitude of the distribution function in the runaway region is then, dominantly,

$$f_R \sim \exp[-\varphi_0^{1}(u_1)] \sim \exp\left[\frac{-E_c}{4E^*} - \left(\frac{M_iE_c}{ME^*}\right)^{1/2}\right].$$

The solution for φ_0 in the runaway region reveals, however, that for small enough a ($a<0.3$, for $M=M_i$) there is a second peak in the distribution function at $u=u_2-a/3$ (for $a\ll 0.4$, $M=M_i$), i.e., just below the upper stagnation point. In the Tokamak case we generally have $4E^* \ll E_c$, so that the runaway ion population would be extremely small. The phenomenon is nonetheless of considerable practical interest in at least two applications:

(I) Neutron production.—For typical Tokamak parameters, the runaway-ion energy lies in the range 10–20 keV, where the cross section for the D-D reaction is very much greater than it is in the range of thermal ion energies. For Maxwellian deuterons at a temperature of 500 eV we have $\langle(\sigma v)\rangle_{\rm DD} \simeq 2\times10^{-24}$ cm^3/sec. The contribution from a fraction f_R of 20-keV runaway deuterons would be $(\sigma v)_{\rm DD} \simeq 7\times10^{-20}f_R$ cm^3/sec: A fraction $f_R \gtrsim 3\times10^{-5}$, roughly corresponding in steady state to $E^*/E_c \gtrsim 0.05$, would contribute more than the thermal distribution. In this context, we note the interesting experimental observation reported in Fig. 7 of Ref. 5, where the "neutron temperature" of a Tokamak discharge in deuterium was found to be enhanced by introduction of auxiliary nonsymmetric magnetic trapping, even though the transverse ion temperature, determined from charge-exchange neutrals, actually decreased (as had been predicted by neoclassical transport theory).

(II) Neutral-beam injection.—One of the most promising techniques for heating the Tokomak discharge beyond the temperatures of a few keV attainable by Ohmic heating is to inject an energetic neutral beam. The most favorable orbits for the resultant trapped energetic ions have velocities predominantly parallel and antiparallel to $\vec{B}$. The neutral-beam technique becomes most powerful at injection energies that are far above T_i; but in order to insure fairly rapid thermalization and to avoid excessive preferential heating of the plasma electrons, one is limited in practice to injection velocities around v_2. Thus one would expect the ion runaway effect to have significant impact on neutral-beam heating in Tokamaks.[9]

We note, finally, that contributions to the Tokamak anomalous resisitivity due to plasma turbulence—for example, through excitation of ion waves—may also create ion runaway conditions. Similarly it is possible that cooperative phenomena may inject superthermal ions at the lower end of the ion runaway region and thus exert a dominant effect on neutron production. The analysis of these phenomena is beyond the scope of the present paper, which is concerned only with classical effects.

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Feynman-Graph Expansion for Critical Exponents*

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The critical exponents γ, η and the "crossover index" φ are computed for generalized classical Heisenberg models with n internal degrees of freedom as an exact expansion in $\epsilon = 4-d$ (d is the number of space dimensions). Results are obtained to order ϵ^2 for γ and to order ϵ^3 for η. The results to this order for the three-dimensional Ising case ($n=\epsilon=1$) are $\gamma=1.244$ and $\eta=0.037$.

In a previous Letter[1] Fisher and the author obtained expansions for critical exponents in powers of $\epsilon = 4-d$, where d is the dimensionality of the system. Generalized Ising and Baxter models were studied using an approximate renormalization-group recursion formula.[2] The results were exact to order ϵ but in error in order ϵ^2.

In this paper exact expansions of critical exponents are reported to order ϵ^2 at least; they were obtained by Feynman-graph techniques. Generalized classical Heisenberg models are discussed; the spin s has n intermal indices. The conventional Heisenberg model corresponds to $n=3$; the Ising model corresponds to $n=1$. The results are as follows:

$$\gamma = 1 + \frac{(n+2)}{2(n+8)}\epsilon + \frac{(n+2)(n^2+22n+52)}{4(n+8)^3}\epsilon^2 + O(\epsilon^3), \tag{1}$$

$$\eta = \frac{(n+2)}{2(n+8)^2}\epsilon^2 + \frac{(n+2)}{2(n+8)^2}\left[\frac{6(3n+14)}{(n+8)^2} - \frac{1}{4}\right]\epsilon^3 + O(\epsilon^4), \tag{2}$$

$$\varphi = 1 + \frac{n\epsilon}{2(n+8)} + \frac{\epsilon^2(n^3+24n^2+68n)}{4(n+8)^3} + O(\epsilon^3), \tag{3}$$

where φ is the "crossover index" of Riedel and Wegner.[3] The numerical results for $\epsilon=1$ ($d=3$) from these series are shown in Table I. The value for φ to order ϵ has previously been obtained by Wegner[6] and by Fisher and Pfeuty.[5]

The Feynman-graph method of this paper is unrelated to the renormalization-group methods of Refs. 1 and 2; in particular the calculation of η described here is distinct from the exact renormalization-group calculation mentioned in Ref. 1. However, a renormalization-group argument will be used below to motivate one step in the Feynman-graph calculation. The calculation of critical exponents in powers of ϵ is simpler in the graphical approach than in the exact renormalization-group approach. The renormalization-group approach remains important for other problems, such as determining the domain of initial Hamiltonians associated with a given set of exponents (see Ref. 1).

The method of calculation will be described briefly. The Hamiltonian used was similar to that of Ref. 1; we define

$$H/kT = \int\{\tfrac{1}{2}r_0 s^2(\vec{x}) + \tfrac{1}{2}[\nabla s(\vec{x}) - \nabla\nabla^2 s(\vec{x})]^2 + u_0 s^4(\vec{x})\}d^d x, \tag{4}$$

where $s(\vec{x})$ is a spin field with n components $s_i(\vec{x})$ [s^2 means $\sum_i s_i^2$ and s^4 means $(\sum_i s_i^2)^2$] and r_0 and u_0 are constants. The term $\nabla\nabla^2 s(\vec{x})$ is present to make integrals converge; its effect is to suppress fluc-