Solution and Critical Behavior of Some "Three-Dimensional" Ising Models with a Four-Spin Interaction

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An exact solution is obtained for some "three-dimensional" Ising models with a four-spin interaction. The spontaneous magnetization is vanishing for all temperatures. The temperature derivative of the susceptibility shows a divergence of the form $(T_c-T)^{-7/8}$ below the critical point. The specific heat shows a logarithmic singularity. Several other related models are introduced with new types of singularities.

Quite recently, Baxter[1] solved the eight-vertex model, which has continuously variable critical exponents. Kadanoff and Wegner,[2] and independently Wu,[3] have pointed out that the above eight-vertex model is equivalent to two two-dimensional Ising models with nearest-neighbor coupling, interacting with one another via a four-spin coupling term.

In the present note, we discuss the critical behavior of a "three-dimensional" Ising model with only a four-spin interaction of the form

$$\mathcal{H} = -\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{L-1}(J_k\sigma_{i,j,k}\sigma_{i,j,k+1}\sigma_{i+1,j,k}\sigma_{i+1,j,k+1}+J_k'\sigma_{i,j,k}\sigma_{i,j,k+1}\sigma_{i,j+1,k}\sigma_{i,j+1,k+1}), \tag{1}$$

and with free ends in the z direction, where $\sigma_{i,j,k}$ denotes an Ising spin (± 1) at the lattice point (i,j,k). As is easily seen, the above model is "anisotropic" in the z direction.

Our solution shows the following properties for the thermodynamic limit ($N, M\to\infty$):

(1) The spontaneous magnetization M_s is vanishing for all temperatures even in the limit $L\to\infty$.

(2) When $J_k=J$ and $J_k'=J'$, the susceptibility χ_0 takes the compact form

$$\chi_0=\frac{m_B^2}{kT}\frac{L(1-C_s^2)-2C_s+2C_s^{L+1}}{L(1-C_s)^2}, \tag{2}$$

where C_s is given by

$$C_s=\pm\left[1-\sinh^{-2}\left(\frac{2J}{kT}\right)\sinh^{-2}\left(\frac{2J'}{kT}\right)\right]^{1/8}. \tag{3}$$

The sign $\pm$ corresponds to the double degeneracy of this system below the critical point. Thus, the temperature derivative of χ_0 shows a singularity of the form

$$(d/dT)\chi_0^{\pm}\sim\mp(T_c-T)^{-7/8}\text{ for } T<T_c. \tag{4}$$

In particular, for L infinite, χ_0 takes the following simple form:

$$\chi_0^{\pm}=\frac{m_B^2}{kT}\frac{1\pm|C_s|}{1\mp|C_s|}. \tag{5}$$

It should be remarked that χ_0^+ diverges exponentially and χ_0^- vanishes exponentially at zero temperature. Below the critical point, spin correlations in the z direction are ferromagnetic in one of the two degenerate states, with a plus sign in (3), and antiferromagnetic in the other, with a minus sign in (3).

(3) The long-range order of this system is expressed by the short-range spin correlation $\langle\sigma_{i,j,k}\times\sigma_{i,j,k+1}\rangle$, which has a singualrity of the form

$$C_s=\langle\sigma_{i,j,k}\sigma_{i,j,k+1}\rangle\sim\pm(T_c-T)^{1/8}. \tag{6}$$

(4) For $J_k=J$ and $J_k'=J'$, two-spin correlations in the z direction are expressed as

$$\langle\sigma_{i,j,k}\sigma_{i,j,k'}\rangle=C_s^{|k-k'|}=(\pm 1)^{|k-k'|}\exp(-|k-k'|/\xi), \tag{7}$$

for $T\leqslant T_c$ and for $1\leqslant k,\ k'\leqslant L$. Thus, the correlation length ξ is given by

$$\xi=(-\ln|C_s|)^{-1}. \tag{8}$$

This vanishes at T_c, proportionally to $[-\ln(T_c - T)]^{-1}$, and is divergent at $T=0$. Then, the derivative of ξ diverges as $\xi \sim [-(T_c - T)\ln^2(T_c - T)]^{-1}$ near T_c. All other two-spin correlations $\langle \sigma_{i,j,k}\sigma_{m,n,k'}\rangle$ vanish identically for all temperatures except the case $i=m$ and $j=n$.

(4) The specific heat shows logarithmic singularities both above and below the critical temperatures. [In general, $L-1$ critical points may appear when all J_k (or/and J_k') differ from one another.]

(5) The partition function of this system is expressed by

$$Z = 2^{MN}\prod_{k=1}^{L-1} Z_2(J_k, J_k'), \tag{9}$$

where Z_2 denotes that of the two-dimensional Ising model,[4] i.e.,

$$Z_2(J,J') = \mathrm{Tr}\exp[\beta \textstyle\sum_{i,j}(J\sigma_{i,j}\sigma_{i+1,j} + J'\sigma_{i,j}\sigma_{i,j+1})]. \tag{10}$$

Consequently, when the values of $\{J_k\}$ and $\{J_k'\}$ are distributed with a probability $\varphi(J_k, J_k')$, the partition function averaged over this distribution function is given by

$$\langle \ln Z\rangle_{\mathrm{av}} = MN\ln 2 + (L-1)\iint \varphi(J,J')\ln[Z_2(J,J')]\,dJ\,dJ'. \tag{11}$$

(6) When one adds to the above Hamiltonian (1) a pair interaction

$$\mathcal{H}_2 = -\sum_{i,j,k} J_k''\sigma_{i,j,k}\sigma_{i,j,k+1}, \tag{12}$$

there occurs no phase transition for $J_k \geqslant 0$ and $J_k' \geqslant 0$.

The above results are all derived easily from the following nonlinear σ-τ transformation, with the abbreviations $\sigma_{i,j,k} = \sigma_k$ and $\tau_{i,j,k} = \tau_k$:

$$\sigma_k = \tau_1\tau_2\tau_3\cdots\tau_k. \tag{13}$$

The inverse relation is given by $\tau_1 = \sigma_1$, $\tau_2 = \sigma_1\sigma_2$, $\cdots$, $\tau_j = \sigma_{j-1}\sigma_j$, $\cdots$, and $\tau_L = \sigma_{L-1}\sigma_L$. Thus, this σ-τ transformation yields a one-to-one correspondence between the configurations $\{\sigma_j\}$ and $\{\tau_j\}$. (In fact, this is a diagonal part of the quantum-mechanical *canonical* σ-τ transformation which has been introduced in discussing time-correlation functions and critical relaxation in a class of one-dimensional stochastic spin systems.[5]) Therefore, for any spin function $f(\{\sigma_j\})$, we have

$$\sum_{\sigma_1=\pm1}\cdots\sum_{\sigma_L=\pm1} f(\{\sigma_j\}) = \sum_{\tau_1=\pm1}\cdots\sum_{\tau_L=\pm1} f(\{\tau_1\tau_2\cdots\tau_j\}). \tag{14}$$

In terms of the σ-τ transformation (13), the Hamiltonian (1) is transformed into $L-1$ independent two-dimensional Ising models of the form

$$\mathcal{H} = -\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=2}^{L}(J_k\tau_{i,j,k}\tau_{i+1,j,k} + J_k'\tau_{i,j,k}\tau_{i,j+1,k}) \tag{15}$$

plus MN free spins $\{\tau_{i,j,1}\}$. Thus, for example, the spin correlation $\langle\sigma_{i,j,k}\sigma_{i,j,k'}\rangle$ for $k<k'$ is calculated as

$$\langle\sigma_{i,j,k}\sigma_{i,j,k'}\rangle = \langle\tau_{ij,k+1}\tau_{ij,k+2}\cdots\tau_{ij,k'}\rangle = C_s^{\,|k-k'|} \tag{16}$$

below the critical point. Otherwise, the spin correlation $\langle\sigma_{ijk}\sigma_{mnk'}\rangle$ for $(ij)\neq(mn)$ is confirmed to vanish.

In connection with the above model, there are several other related systems which can be discussed rigorously with use of the σ-τ transformation (13).

(a) When a next-nearest-neighbor interaction of the form

$$\mathcal{H}_4' = -\lambda\sum\sigma_{i,j,k}\sigma_{i,j,k+2}(\sigma_{i+1,j,k}\sigma_{i+1,j,k+2} + \sigma_{i,j+1,k}\sigma_{i,j+1,k+2}) \tag{17}$$

is added to the Hamiltonian (1), *continuously variable critical exponents* appear, as in Baxter's solution.[1] In order to explain this peculiar critical behavior, we apply the σ-τ transformation to the Hamiltonian $\tilde{\mathcal{H}} = \mathcal{H} + \mathcal{H}_4'$, which results in

$$\tilde{\mathcal{H}} = -\sum_{\langle ij;mn\rangle}\Big(\sum_{k=2}^{L} J\tau_{ij,k}\tau_{mn,k} + \lambda\sum_{k=2}^{L-1}\tau_{ij,k}\tau_{ij,k+1}\tau_{mn,k}\tau_{mn,k+1}\Big), \tag{18}$$

where $\langle ij; mn\rangle$ denotes the summation over all nearest-neighbor lattice sites in two dimensions. Here we may be satisfied in showing that critical exponents defined in terms of τ spins vary continuously as the parameter λ does. For example, we discuss the singularity of the susceptibility χ_0 defined by

$$\chi_0 = (m_B^2/kT)\sum_{\vec{R}, \vec{R}'}\langle \tau_{\vec{R}}\tau_{\vec{R}'}\rangle, \tag{19}$$

with $\vec{R} = (i, j, k)$. We use first-order perturbation theory following Kadanoff and Wegner.[2] It is easily shown that

$$(\partial\chi_0/\partial\lambda)_{\lambda=0} = u(\partial\chi_0/\partial J)_{\lambda=0}, \tag{20}$$

where u denotes the short-range correlation $\langle\tau_{i,j,k}\tau_{i,j+1,k}\rangle_{\lambda=0}$ $[\simeq u_c + a\epsilon(0)\ln\epsilon(0);\ a = 4J/\pi kT_c]$ and for simplicity we have assumed that $J_k = J_k' = J$. Now the susceptibility may take the asymptotic form

$$\chi_0 \sim [\epsilon(\lambda)]^{-\gamma(\lambda)}, \tag{21}$$

near the critical point $T_c(\lambda)$, where

$$\epsilon(\lambda) = [T - T_c(\lambda)]/T_c(\lambda). \tag{22}$$

Differentiating (21) with respect to λ, we have

$$(\partial\chi_0/\partial\lambda)_{\lambda=0} \simeq -\gamma(0)\chi_0\epsilon'(0)/\epsilon(0) - [\ln\epsilon(0)]\chi_0\gamma'(0). \tag{23}$$

On the other hand, from the relation (20), $(\partial\chi_0/\partial\lambda)_{\lambda=0}$ should take the form

$$\left(\frac{\partial\chi_0}{\partial\lambda}\right)_{\lambda=0} \simeq -\gamma(0)u_c\left(\frac{\partial\epsilon}{\partial J}\right)_0\frac{\chi_0}{\epsilon(0)} - \gamma(0)a\epsilon(0)[\ln\epsilon(0)]\left(\frac{\partial\epsilon}{\partial J}\right)_0\frac{\chi_0}{\epsilon(0)}. \tag{24}$$

Consequently, by comparing Eqs. (23) and (24), one obtains the relations

$$T_c'(0) = T_c u_c J^{-1} = T_c(\sqrt{2}\,J)^{-1}, \tag{25}$$

$$\gamma'(0) = a\gamma(0)\left(\frac{\partial\epsilon}{\partial J}\right)_0 = a\gamma(0)\left(-\frac{1}{J}\right) = -\frac{\gamma(0)}{kT_c}\frac{4}{\pi}. \tag{26}$$

Therefore, $\gamma(\lambda)$ takes the form

$$\frac{\gamma(\lambda)}{\gamma(0)} = 1 - \frac{4}{\pi}\frac{\lambda}{kT_c} + O(\lambda^2). \tag{27}$$

In the same way, we have

$$\frac{\beta(\lambda)}{\beta(0)} = \frac{\nu(\lambda)}{\nu(0)} = \frac{\gamma(\lambda)}{\gamma(0)} \tag{28}$$

up to the first order of λ. This λ dependence agrees with that obtained by Kadanoff-Wegner in two dimensions, although the present model (18) is three-dimensional and consequently the λ dependence of higher order may be different from theirs. The λ dependence (27) is also expected to be valid for the susceptibility defined in terms of original σ spins.

(b) It is also possible to solve n-dimensional Ising models with a 2^n-spin interaction defined by

$$\mathcal{H}_m = -\sum_{\langle\vec{R}_1\vec{R}_2\cdots\vec{R}_m\rangle} J(\vec{R}_1, \vec{R}_2, \cdots\vec{R}_m)\sigma(\vec{R}_1)\sigma(\vec{R}_2)\cdots\sigma(\vec{R}_m), \tag{29}$$

where $m = 2^n$, and $\langle\vec{R}_1\vec{R}_2\cdots\vec{R}_m\rangle$ denotes a set of lattice points located at the corners of each unit cell in the n-dimensional lattice. The Hamiltonian (29) can be reduced to a lot of independent one-dimensional Ising models, with nearest-neighbor pair interaction, by applying the σ-τ transformation (13) $n-1$ times successively. Consequently there occurs no phase transition.

(c) A solution is obtained for a two-dimensionsl Ising model with two- and four-spin interactions of the form

$$\mathcal{H}_{2,4} = -J\sum_{ij}\sigma_{i,j}\sigma_{i,j+1}\sigma_{i+1,j}\sigma_{i+1,j+1} - J'\sum_{ij}\sigma_{i,j}\sigma_{i+2,j}, \tag{30}$$

with use of the σ-τ transformation (13). The specific heat shows a logarithmic singularity near the critical point.

Several other related examples will be reported in a separate paper[6] with detailed arguments on the above models. [As should be the case, spin correlations obtained exactly in the model (1) satisfy Griffiths-Kelly-Sherman inequalities[7] for $J_k \geq 0$ and $J_k' \geq 0$.]

[1]R. J. Baxter, Phys. Rev. Lett. 26, 832 (1971), and to be published.
[2]L. P. Kadanoff and F. J. Wegner, Phys. Rev. B 4, 3989 (1971).
[3]F. Y. Wu, Phys. Rev. B 4, 2312 (1971).
[4]L. Onsager, Phys. Rev. 65, 117 (1944).
[5]B. U. Felderhof and M. Suzuki, Physica (Utrecht) 56, 43 (1971).
[6]M. Suzuki, to be published.
[7]R. B. Griffiths, J. Math. Phys. 8, 478, 484 (1967); D. G. Kelly and S. Sherman, J. Math. Phys. 9, 466 (1968); J. Ginibre, Phys. Rev. Lett. 23, 828 (1969).

β-Delayed Proton Emission of ^{23}Al†

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The weak β-delayed proton emitter ^{23}Al, with a half-life of 470 ± 30 msec, was produced by the reaction ^{24}Mg$(p, 2n)^{23}$Al. We observed delayed protons with a center-of-mass energy of 870 ± 30 keV and a maximum production cross section ≈ 220 nb.

Recent mass measurements[1] have shown ^{23}Al to be the lightest, nucleon-stable member of the mass series $A = 4n+3$, $T_z = \frac{1}{2}(N-Z) = -\frac{3}{2}$; however, no technique capable of characterizing the decay properties of these nuclides has been demonstrated. Using the ^{24}Mg$(p, 2n)$ reaction we have observed ^{23}Al through its β-delayed proton emission. Extension of this approach to heavier $T_z = 0$ target nuclei should, in principle, permit the observation of several heavier members of this mass series which are predicted[2] to be nucleon stable (^{27}P through ^{35}K).

The external proton beam of the Berkeley 88-in. cyclotron was used to induce the reaction ^{24}Mg$(p, 2n)^{23}$Al on 99.96%-enriched ^{24}Mg targets. Two independent experimental approaches were used. In the first of these, delayed protons from activity in the target were detected in a counter telescope mounted downstream from the target behind a slotted, rotating wheel. This wheel controlled the duration of the beam pulse and shielded the detectors during the beam-on intervals. Beam pulsing was achieved by modulating the cyclotron dee voltage; we utilized beam intensities of up to 8 μA on target. In these experiments a detector telescope, consisting of an 8-μm ΔE detector, fed a Goulding-Landis particle identifier. Any long-range particles were eliminated by a 50-μm reject detector. In order to observe low-energy protons (and α particles), singles spectra were recorded from the 8-μm detector as well as from an additional 14-μm detector. All detectors (except the ΔE) were cooled to $-25\,^\circ$C. Accurate energy scales were obtained in this setup by scattering, from a thin Au foil, H_2^+ beams of 0.63 and 1.15 MeV/nucleon as measured in an analyzing magnet (a 4-μm ΔE detector was used for this calibration).

The second experimental configuration employed a helium-jet system[3] which swept nuclei recoiling from the target through a 0.48-mm-diam, 80-cm-long capillary and deposited them on a 550-μg/cm^2 Ni collector foil. At 1.2-sec intervals this foil was quickly (~ 25 msec) moved by a solenoidal stepping motor from the collection position to a position in front of a counter telescope. The telescope and its associated electronics were identical to those in the first setup except that it employed a 6-μm ΔE detector. In these experiments we utilized a continuous proton beam of up to 8 μA on target. By comparing the yields obtained in both experimental configurations (corrected for recoil-range effects), the absolute efficiency of the helium-jet technique for collecting ^{23}Al was determined to be ~10%. This disadvantage was offset by the higher attain-