the scalars associated with the stress tensor for the field ψ. Thus Q is a physical scalar, and must be bounded on the (nonsingular) horizon:

$$\psi^2 W^{-1}(\psi_{,\rho}{}^2+\psi_{,z}{}^2) \text{ bounded on horizon.} \tag{6}$$

Now, Schwarz's inequality gives us for the boundary term of (5)

$$[\psi W^{-1}(\psi_{,\rho}n_\rho+\psi_{,z}n_z)]^2 \leq \psi^2 W^{-2}(\psi_{,\rho}{}^2+\psi_{,z}{}^2)(n_\rho{}^2+n_z{}^2). \tag{7}$$

If we now take into account (2), (6), and the fact that $d\sigma$ is nonsingular on the horizon, we see from (7) that the left-hand side of (5) vanishes.

The integrand of the right-hand side of (5) is positive definite (recall that W is non-negative) everywhere in the black-hole's exterior. It follows that the integral can vanish only if ψ vanishes identically in the black-hole's exterior. Thus a rotating black hole in its final state cannot be endowed with an exterior scalar meson field, a π meson field for example. The proofs for vector and spin-2 fields are more complicated than the above, but the basic idea is the same.[10] For nonrotating (static) black holes, axial symmetry is not guaranteed by Hawking's theorem, so one must assume the most general static metric; but this causes no particular problems.[10]

The author wishes to thank Robert M. Wald and Professor John A. Wheeler for many helpful suggestions and comments.

*Work supported in part by the U. S. Air Force Office of Scientific Research under Grant No. AF49(638)-1545, and by the National Science Foundation, under Grant No. GP 30799X.

†National Science Foundation Predoctoral Fellow.

[1]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1971).

[2]W. Israel, Phys. Rev. 164, 1776 (1967), and Commun. Math. Phys. 8, 245 (1968).

[3]B. Carter, Phys. Rev. Lett. 26, 331 (1971).

[4]S. W. Hawking, to be published.

[5]R. M. Wald, Phys. Rev. Lett. 26, 1653 (1971), and to be published.

[6]J. B. Hartle, Phys. Rev. D 3, 2938 (1971).

[7]C. Teitelboim, to be published.

[8]J. A. Wheeler, in *Cortona Symposium on Weak Interactions*, edited by L. Radicati (Accademia Nazionale Dei Lincei, Rome, 1971).

[9]For an early discussion of transcendence see B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (Univ. of Chicago Press, Chicago, 1965), p. 143.

[10]J. D. Bekenstein, Phys. Rev. D (to be published), and to be published.

[11]Y. Takahashi, An Introduction to Field Quantization (Pergamon, New York, 1969), pp. 48–54.

[12]Ya. B. Zeldovich and I. D. Novikov, *Stars and Relativity* (Univ. of Chicago Press, Chicago, 1971), p. 77.

[13]R. Arnowitt, S. Deser, and C. Misner, Phys. Rev. 120, 313 (1960).

[14]C. Teitelboim, to be published.

Energy Straggling and Radiation Reaction for Magnetic Bremsstrahlung*

C. S. Shen and D. White
Department of Physics, Purdue University, Lafayette, Indiana 47907
(Received 13 December 1971)

When the average energy of the photon emitted by synchrotron radiation becomes appreciable compared to the energy of the particle, the particle will undergo straggling in its energy loss, which in turn broadens the radiation spectrum. The energy distributions of particles and the emitted photons are calculated using the method of quantum electrodynamics. The results are presented together with effects due to classical radiative reaction for experimental test. The significance of energy straggling in astrophysics is discussed briefly.

In ordinary circumstances the motion and radiation of a charged particle in a magnetic field can be adequately described by classical electrodynamics neglecting the radiation damping. The significance of the radiation effect is predicted by the parameter $R_c = \frac{2}{3}\gamma^2 H/H_c$, which represents the ratio of the radiative reaction force to the Lorentz force. Here $H_c = m^2c^4/e^3 = 6\times 10^{15}$ G and $\gamma = E/mc^2$. On the other hand, the validity of classical radiation theory is limited by the condition $R_q = \frac{3}{2}\gamma H/H_q \ll 1$, where $H_q = m^2c^3/e\hbar = 4.4\times 10^{13}$ G. The ratio of the mean energy of the emitted photons to the energy of the particle is $\sim R_q(1+R_q)^{-1}$. Therefore quantum recoils become significant at $R_q \gtrsim 1$, and for a consistent treatment one has to use the method of quantum electrodynamics.[1]

Quantum and radiative corrections on synchrotron emission had stimulated considerable interest in astrophysics,[2,3] where in many cases R_c, R_q, or both have large values. Recent developments on the technique of flux compression make possible the generation of transient magnetic fields up to 10 MG in the laboratory.[4] Also, electron beams of energy up to a few hundred GeV should be available soon from the National Accelerator Laboratory. One can readily see that from these parameters $R_c \gtrsim 100$ and $R_q \gtrsim 0.1$. For the first time, then, it appears that both the classical Lorentz-Dirac equation and the effect of quantum electrodynamics in an external field can be tested experimentally by the combination of high-energy accelerators and megagauss targets. A series of such experiments, using megagauss pulses as targets for high energy electron beams (abbreviated as MPEB hereafter), have already been carried out by Herlack *et al.*[4] and Erber[5] at the Stanford Linear Accelerator Center.

In a previous article,[3] one of us had analyzed the strong radiative reaction effects for a classical relativistic particle. In this Letter we shall first outline the classical radiative reaction effects which may be tested in the type of experiments mentioned above, then we shall present a detailed quantum mechanical calculation. Implications of the present work in astrophysics will be discussed briefly.

(1) Classical radiative corrections.—The trajectory of a particle in a plane perpendicular to $\vec{H}$ at strong radiative damping is derived in Ref. 3. It can be seen easily from Eq. (5) of Ref. 3 that the deflection of a particle of initial energy E_0 after traversing a distance L in magnetic field H is[6]

$$\theta = f(L/c) = (1 + R_c eHL/2E_0)eHL/E_0, \tag{1}$$

i.e., in addition to the normal deflection $\theta_0 = eHL/E_0$ there is an additional, radiation-induced deflection $\delta\theta_c = R_c\theta_0^2/2 = e^5H^3L^2/3m^4c^8$. This additional shift corresponds to the shrinking of the orbit of the particle.

An interesting feature of the deflection angle is that the magnitude of $\delta\theta_c$ is inversely proportional to the fourth power of mass, where the normal deflection is a function of energy only. Thus a beam of μ^- or π^- sent through the magnetic pulse simultaneously with the electron beam can serve the purpose of calibration. There are of course other radiative reaction effects which can be studied in MPEB, but from an experimental point of view none of them is as distinctive as the deflections. All these results will be modified by quantum effects for the National Accelerator Laboratory electron beams.

(2) Quantum mechanical treatment.—The quantum theory of synchrotron radiation has been studied extensively in the past.[7] The energy eigenvalue associated with the Dirac wave function depends only on the total quantum number $\boldsymbol{n}$: $E_n = mc^2(1 + 2nH/H_q)^{1/2}$ ($n \gg 1$ for relativistic particles). The transition rate between two highly relativistic energy levels, after summing over the degenerate states, is

$$\lambda_{ij} = \alpha\omega_H f(y)/\sqrt{3}\,\pi\gamma_i^2\gamma_j, \tag{2}$$

where

$$f(y) = \int_y^\infty K_{5/3}(x)\,dx + R_q^2y^2(1+R_qy)^{-1}K_{2/3}(y),$$
$$y = (E_i - E_j)/R_{qi}E_j, \quad \alpha = e^2/\hbar c, \text{ and } R_{qi} = \gamma_i H/H_q.$$

The radiation spectrum of an electron at energy E_i is given by[8]

$$I_i(\omega)\,d\omega = \sum_j \hbar\omega_{ij}\lambda_{ij} = \frac{3\sqrt{3}}{4\pi}c\gamma^2H^2\left(\frac{e^2}{mc^2}\right)^2\frac{y^2}{(1+R_qy)^3}f(y)\frac{d\omega}{\omega(1-\hbar\omega/\gamma mc^2)}, \tag{3}$$

where E_j is replaced by $E_i - \hbar\omega$ in y.

Equation (3) is the usual quantum mechanical formula for synchrotron radiation, which reduces to the classical form at $R_{qi} \to 0$. $I_i(\omega)$, however, is not the radiation spectrum one will observe for strong radiation, where quantum broadening leads to a fast spread in energy of particle states. This spread is analogous to the energy straggling when a fast particle travels through a thickness of matter where quanta emitted through bremsstrahlung carry away energy comparable to that of the particle. In synchrotron radiation, the transition rate λ_{ij} is a flat function of ΔE_{ij} up to $\Delta E_{ij} \sim R_{qi}(1+R_{qi})^{-1}E_i$; thus, the energy-loss process can be considered as cascade transitions with random steps of maximum magnitude $R_q(1+R_q)^{-1}$. The statistical fluctuation is negligible only if $R_q \ll 1$. As we shall see, the quantum broadening which is significant even at relatively small values of R_q dominates all other quantum corrections. A beam of monoenergetic electrons injected into an intense magnetic field will soon spread over a large range of energy. This statistical nature of deceleration, which implies that a portion of particles will remain in an energy range well above that predicted by the classical formula [Eq. (4) of Ref. 3], is not without interest in astrophysics. The energy straggling also modifies the radiation spectrum. Since the observation time of a photon detector is usually much longer than τ, it receives not the instantaneous spectrum, which represents the distribution of photons emitted from electrons at a given energy level, but a cumulative spectrum including photons emitted from all subsequent transitions in the cascade process.

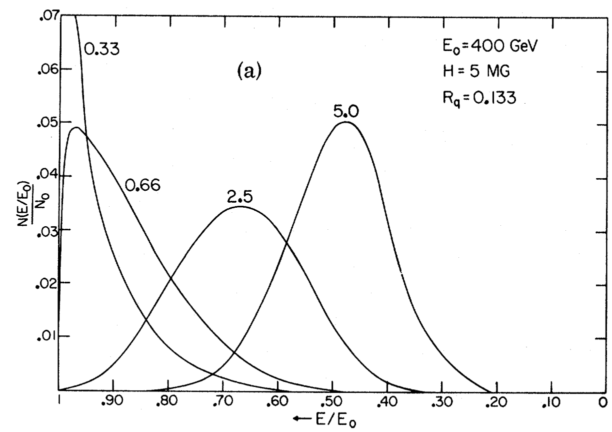
We shall illustrate these effects by a quantative analysis aimed at direct verification in the MPEB experiments. Let $\rho_i(t)$ be the number of electrons in state i. We have[9]

$$\frac{d\rho_i}{dt}=-\frac{1}{\tau_i}\rho_i+\sum_{j>i}\lambda_{ji}\rho_j,\quad i=1,2,\cdots. \tag{4}$$

Since the states are nearly continuous for relativistic energy, this set of ordinary differential equations may be replaced by a single partial differential equation,

$$\frac{\partial\rho(E,t)}{\partial t}=-\frac{1}{\tau(E)}\rho(E,t)+\int_E^\infty\lambda(E,E')\rho(E',t)dE'. \tag{5}$$

Equation (5) has been studied in great detail in shower problems. Eyges[10] has shown that it leads to the Bethe-Heitler formula[11] for energy straggling in the case of bremsstrahlung, for which the radiation cross section can be approximated by a simple expression with maximum photon energy equal to the electron energy. In the case of synchrotron emission, however, because of the nonlinear dependence of λ on E we were unable to find an analytic solution for Eq. (5) with adequate accuracy. Instead, Eq. (5) is solved by a straightforward numerical method.[9] The solution $\rho(E, t)$ obtained describes the energy distribution of the particles at time t with initial distribution $\rho(E, 0)$. Other observables such as the expected instantaneous radiation spectrum can be readily calculated from $\rho(E, t)$ by $I(\omega, t)=\int\rho(E, t)\times I(\omega, E)dE$. For illustration we have plotted $\rho(E, t)$ in Fig. 1 for $R_q=0.133$ ($E_0=400$ GeV, $H=5$ MG) and $R_q=0.655$ ($E_0=1000$ GeV and $H=10$ MG) and $\rho(E,0)=\delta(E-E_0)$. The difference between $\gamma_c(t)$, the classical energy, and $\langle\gamma_q(t)\rangle$, the average quantum mechanical energy, can be readily explained by the reduction of the radiation rate due to the quantum mechanical effect. The spread in energy space is rather phenomenal. Within a distance of less than 1% of the initial radius of gyration, the half-width in energy becomes comparable to the average energy. Obviously, the maximum spread decreases with the decrease of R_q (varies as $R_q^{1/2}$). A quantity of special interest in astrophysics is the probability that an electron of initial energy E_0 has retained an energy greater than E after having spent a time t in a magnetic field H. We find, by a crude estimation, that for $R_q \gtrsim 1$ there are 10% of particles with energy greater than $0.02E_0$ when the "pre-



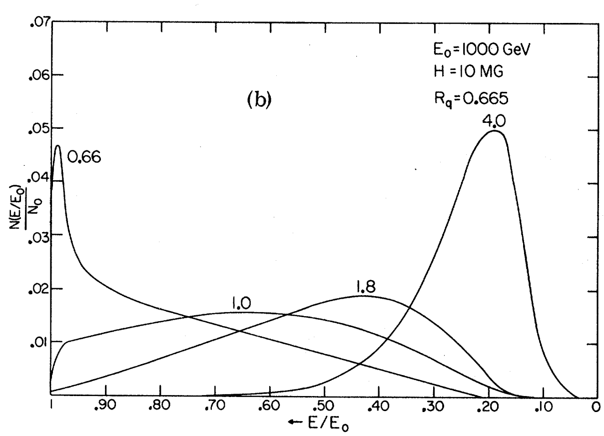


FIG. 1. Energy spread. The numbers affixed to the curves are the magnetopath lengths LH in units of cm MG. The classical energy [calculated from Eq. (4) of Ref. 3] and the expected quantum mechanical energy (obtained by averaging over the distribution) are, respectively, $\gamma_c/\gamma_0=0.92$, 0.86, 0.61, and 0.44 and $\langle\gamma_q\rangle/\gamma_0=0.95$, 0.90, 0.68, and 0.50 for the four curves in (a), and $\gamma_c/\gamma_0=0.55$, 0.41, 0.31, and 0.16 and $\langle\gamma_q\rangle/\gamma_0=0.77$, 0.65, 0.51, and 0.25 for the four curves in (b).

dicted" average energy is $0.001E_0$.

A comparison may be made between the present case of energy straggling and that of betatron oscillations.[7] The former represents random fluctuations caused by the downward cascade transitions between states of different energies. The latter occurs at much lower energy $[E \sim mc^2 \times (H/H_q)^{-1/2}]$, where the effects of radiation loss on particle energy are negligible. Thus the betatron effect results in no fluctuations in energy space. But recoils from emitted photons can still cause changes in the radial quantum number, which lead to a fluctuation of the orbital center.

The instantaneous radiation spectrum $I(z, t)$ ($z = \frac{2}{3}\omega/\gamma_0^2\omega_H$) for the electron beam after traversing a path length L in a field H is shown in Fig. 2. This spectrum is equivalent to the expectation radiation spectrum of an electron of initial energy γ_0 after spending a time $t = L/c$ in the magnetic field. For comparison, the usual quantum mechanical synchrotron spectrum corresponding to $\langle\gamma_q(t)\rangle$ is also plotted. For most cases of practical interest, however, the particles are trapped in the magnetic field and the observation time is much longer than both the gyration period and the lifetime of the particle. The observed spectrum is the cumulative spectrum $I_c(z, \gamma_0) = T^{-1} \times \int_0^T I(z, t)\, dt$ instead of the instantaneous spectrum $I(z, t)$. In Fig. 2 we have plotted $I_c(z, \gamma_0)$ for a relatively short T ($LH = 1.8$ cm MG, the path length corresponding to half the initial energy

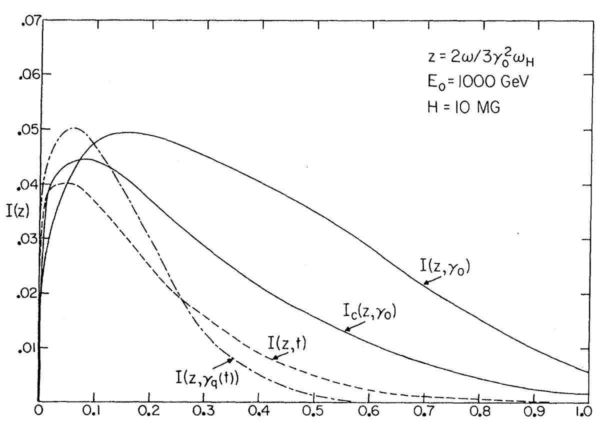


FIG. 2. Radiation spectrum in units of $\gamma_0^2 e^4 H^2/m^2c^3$. $I_c(z, \gamma_0)$ is the cumulative spectrum of a particle of initial energy γ_0 after having traversed 1.8 cm MG; $I(z, \gamma_q(t))$ is the instantaneous spectrum of a particle with energy $\langle\gamma_0(t)\rangle$, the average energy after having traversed 1.8 cm MG; $I(z,t)$ is the actual instantaneous spectrum including quantum broadening; $I(z, \gamma_0)$ is the instantaneous spectrum of a particle of energy γ_0.

loss at $R_q = 0.655$). Compared with $I(z, \gamma)$, the maximum of $I_c(z, \gamma_0)$ shifts to smaller z because of the contribution from soft photons emitted during the cascade transitions between low energy levels. The slope of the spectrum between the new and the old maxima is $\sim\omega^\alpha$, with $0 > \alpha(R_q) > -\frac{1}{2}$. The longer the T, the more significant the contribution from low-energy-level transitions, and the further the shift of the maximum. At $T \to \infty$ (i.e., the particle loses all of its energy during the observation), I_c becomes similar to the classical cumulative spectrum (Fig. 2 of Ref. 3), but the spectral index is larger than $-\frac{1}{2}$ at low frequency and the dip at $z \lesssim 1$ is flattened somewhat due to the spread of the particle in energy space.

The radius of curvature of a particle in a magnetic field is proportional to energy, hence spread in energy space also leads to a spread in the deflection angle. For $R_q \leq 1$ where quantum corrections other than the spread are small, the root mean square fluctuation $\Delta\theta = (\langle\theta^2\rangle - \langle\theta\rangle^2)^{1/2}$ can be adequately described by $\Delta\theta \cong 2(1 - \langle E\rangle/E_0) \times N^{-1/2}\langle\theta\rangle$, where $N = (5/2\sqrt{3})(H/H_c)(Lmc/\hbar)$ is the average number of photons emitted along L. The ratio of the "shift" due to radiation reaction to the spread due to quantum broadening is

$$\delta\theta_c/\Delta\theta = 0.62(1 + 0.083H^2LE_0)(LH)^{1/2}, \tag{6}$$

where L, H, and E_0 are in units of centimenters, megagauss, and ergs, respectively. The factor $(LH)^{1/2}$ appears because the radiation reaction effect is cumulative, i.e., $\delta\theta_c/\theta \propto LH$, while the quantum broadening is essentially a random walk process, i.e., $\Delta\theta \propto (LH)^{1/2}$. From an experimental point of view it is essential to extend the path length in order to obtain an unambigous measurement of $\delta\theta_c$.

We would like to thank Professor T. Erber for many stimulating discussions.

*Work supported in part by Purdue Research Foundation and National Aeronautics and Space Administration Contract No. 4847-52-13969.

[1]See, for example, T. Erber, Rev. Mod. Phys. 38, 626 (1966).

[2]See, for example, F. Pacini, *Neutron Stars, Pulsar Radiation, and Supernova Remnants* (Gordon and Breach, New York, 1971).

[3]C. S. Shen, Phys. Rev. Lett. 24, 410 (1970).

[4]F. Herlach, R. McBroom, T. Erber, J. Murry, and R. Gearhart, to be published.

[5]T. Erber, "Some External Problems in Quantum Electrodynamics," in Internationale Universitätswochen für Kernphysik, Karl-Franzers-Universität Graz:

Tenth Schladming Winter School in Physics, 1971 (unpublished).

[6]This deflection was also derived by H. G. Latal (private communication.

[7]A. A. Sokolov and I. M. Ternov, *Synchrotron Radiation* (Akademei, Berlin, 1968).

[8]J. Schwinger, Proc. Nat. Acad. Sci. U.S. 40, 132 (1954).

[9]Induced transitions are neglected here. A more elaborate calculation including this effect is underway (D. White, to be published). A detailed description of the numerical method used in solving Eq. (5) will also be included there.

[10]L. Eyges, Phys. Rev. 76, 264 (1949).

[11]H. A. Bethe and W. Heitler, Proc. Roy. Soc., London 146, 83 (1934).

Inclusive N-Body Reaction in Nucleon-Nucleon Interactions at Ultrahigh Energies*

P. L. Jain and Z. Ahmad

High Energy Experimental Laboratory, Department of Physics, State University of New York at Buffalo, Buffalo, New York 14214

(Received 15 December 1971)

Inclusive reactions for multiparticles produced from cosmic-ray emulsion data at $\sim 10^{12}$ eV are presented and are compared with the predictions of limiting-fragmentation distribution.

Within the last couple of years a number of papers both theoretical[1-3] and experimental[4] have appeared on the inclusive reactions of type $A+B \rightarrow C+$anything as a source of information for understanding the mechanism of multiparticle production processes. Benecke *et al.*[1] introduced the hypothesis of limiting fragmentation, and Feynman[2] predicted the behavior for the function $f(p_t, x)=(2E/\pi s^{1/2})\, d^2\sigma/dx\, dp_t^2$, where E is the energy of the observed particle in the c.m. system, the Feynman variable $x=2p_l/s^{1/2}$, and s the square of the total energy in the c.m. system. It can be shown that these two hypotheses are equivalent at high energies.[5]

Measuring an angle and the momentum of a single particle has been rather popular, and practically all the experiments reported recently have been for a single observed particle (inclusive one-body reaction) emerging from high-energy hadron-hadron interactions. A single particle gives a rather incomplete reflection of the actual behavior of general collisions. Moreover, because only one final particle is detected in these experiments, there is no empirical way to ascertain whether the observed particle is scattered directly or is the decay product of a resonance. If it is from a resonance, a different approach is desirable.

Most of the data presented so far on single-particle production have been taken from the accelerator energy range. Recently, charged-particle multiplicities and the angular distribution of the secondary particles in pp interactions[6] for the energy range 100–800 GeV have been reported from cosmic-ray experiments. According to these authors, their angular distribution was biased against smaller angles, ≤ 2 mrad, and larger angles, $>40°$.

We present here an analysis of 21 cosmic-ray jets[7-12] of energy $\sim 10^{12}$ eV, which were selected out of about 135 jets with ≤ 4 black prongs. The complete analysis of a jet observed in nuclear emulsion is rather tedious[7] because of the many parameters that are required for its analysis. The energy of the secondary particles were determined by multiple and relative scattering measurements. We know of no published data reporting an inclusive n-body reaction with jets studied completely in emulsion of TeV energy range. The data presented here have energy ranges between 1–100 TeV and average multiplicity ~ 17. It is known that most of the particles produced in high-energy hadron collisions are pions. Production of strange particles is less abundant, and baryon-antibaryon pair production is even more rare.

The differential cross section for a single inclusive process, i.e., $N+N\rightarrow$ (one identified particle) +anything, is written as

$$\begin{aligned} d\sigma &= dp^3 f(p_t, p_l, s)/E \\ &= \pi\, d(p_t^2)\, dp_l f(p_t, p_l, s)/E \end{aligned} \tag{1}$$

where E is the energy and p_t and p_l are the transverse and longitudinal momenta of the observed particle in the c.m. system. Feynman has sug-