rotation in phase space. Therefore, the oscillations in $\delta\Gamma$, Fig. 1, occur at a frequency near ω_B. The oscillations in $\delta\Omega$, Fig. 2, take place at essentially twice this rate because they reflect changes in kinetic energy, which is an even function of velocity.

We have also solved the problem for the experimentally more realistic case where the frequency is fixed and the wave number exhibits nonlinear oscillations. For this case, the nonlinear shift in wave number is given by $\delta k = \delta\Omega(k_B x)/(-v_g)$, where v_g is the wave group velocity and $k_B \equiv \omega_B/v_p$ is the bounce wave number. The functional dependence of $\delta\Omega(k_B x)$ is given in Fig. (2), with $k_B x$ replacing $\omega_B t$. To illustrate the feasibility of measuring such a wave-number shift, we note that for typical experimental parameters[3] ($v_p/v_g = 2.5$, $v_p/v_T \simeq 4$, $\omega_B/kv_p = 0.08$) one obtains an asymptotic shift of 7 rad/m. The spatial oscillation in phase has an amplitude of about 1 rad. These quantities should be measurable.

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[11]The operator $(2\gamma_L - d/dt)$ appears on the right-hand side of Eq. (11) because our formulation is in terms of $\delta\Gamma$, rather than the full damping coefficient.

[12]The "radiation pressure" calculated by W. E. Drummond [Phys. Fluids 7, 816 (1964)] produces a change in the plasma density that also gives rise to a wave-number shift. We find this to be of order $\delta k/k \sim (\omega_B/\omega)^4(v_p/v_T)^2$, which is smaller than the effect we calculate. Also, in a real experiment one must make sure that the large-amplitude wave does not change the density by such indirect methods as modification of boundary conditions or injection regions.

Magnetic Pressure of Solid ^{3}He

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The pressure due to the magnetic ordering in solid ^{3}He is calculated using a Hamiltonian which includes triple exchange. This added degree of freedom supplies a simple explanation for the observed behavior of the pressure as a function of temperature.

The measurement of the pressure change as a function of temperature in solid ^{3}He was first used by Panczyk and Adams[1] (PA) to obtain a value for J_2, the pair-exchange frequency. Three groups of experimentalists[2-4] have recently repeated these measurements in the presence of a large external magnetic field. The results of the three experiments disagree quantitatively from the result expected on assuming solid ^{3}He to be a "good" spin-$\frac{1}{2}$ Heisenberg solid.[5] The Heisenberg Hamiltonian includes only the possibility of pair interactions, which for ^{3}He are antiferromagnetic and on the order of 1 mK. Triple exchange has been shown by Thouless[6] to be inherently ferromagnetic. The relative magnitude of triple exchange compared with pair exchange has been estimated by Guyer and Zane[7] (GZ). This paper will include triple exchange in the Hamiltonian used to calculate the pressure due to spin ordering. The results we obtain are substantially in agreement with the experiment of Kirk and Adams[3] (KA).

The usual Hamiltonian written for solid ^{3}He in an external magnetic field is

$$\mathcal{H} = \mathcal{H}_{ex}^{(2)} + \mathcal{H}_Z, \tag{1}$$

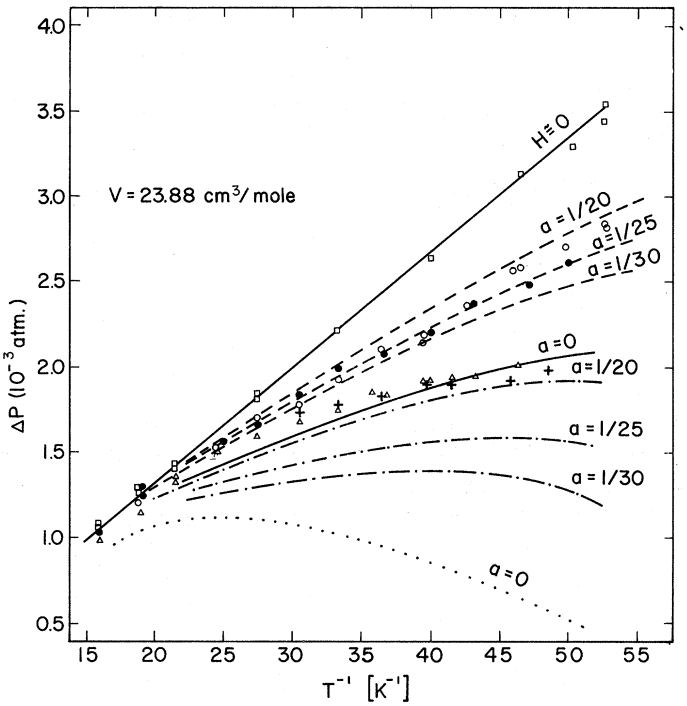

FIG. 1. The change in pressure as a function of inverse temperature. For $H \simeq 0$: squares, experiment by KA; solid line, theory without triple exchange. For $H = 40$ kG: circles, experiment by KA; dashed lines, theory with triple exchange; solid line, theory without triple exchange ($a = 0$). For $H = 60$ kG: triangles and crosses, experiment by KA; dot-dashed lines, theory with triple exchange; dotted line, theory without triple exchange ($a = 0$).

where $\mathcal{H}_{ex}^{(2)}$ is the Heisenberg spin Hamiltonian,

$$\mathcal{H}_{ex}^{(2)} = -2\hbar J_2 \sum_{i<j} \vec{\sigma}_i \circ \vec{\sigma}_j , \tag{2}$$

and $\mathcal{H}_Z$ is the Zeeman Hamiltonian,

$$\mathcal{H}_Z = -\gamma\hbar \vec{H} \circ \textstyle\sum_i \vec{\sigma}_i , \tag{3}$$

$\vec{\sigma}_i$ being the angular momentum of the ith atom. The system is ferromagnetic if $J_2 > 0$ and antiferromagnetic if $J_2 < 0$. The magnetic moment of ^{3}He is written as $\gamma\hbar$. When this Hamiltonian is used to calculate the magnetic pressure, there is a sizable disagreement between theory and experiment (see Fig. 1). We include the next higher-order piece in the exchange Hamiltonian, triple exchange, which can be written as[6]

$$\mathcal{H}_{ex}^{(3)} = 2\hbar J_3 \sum_{i<j<k} (\vec{\sigma}_i \circ \vec{\sigma}_j + \vec{\sigma}_j \circ \vec{\sigma}_k + \vec{\sigma}_k \circ \vec{\sigma}_i), \tag{4}$$

where J_3 is the frequency with which three particles cyclically permute. The triple is ferromagnetic when $J_3 < 0$ and antiferromagnetic when $J_3 > 0$. The exchange Hamiltonian is now written as

$$\mathcal{H}_{ex} = \mathcal{H}_{ex}^{(2)} + \mathcal{H}_{ex}^{(3)} . \tag{5}$$

We substitute $\mathcal{H} = \mathcal{H}_{ex} + \mathcal{H}_Z$ into the partition function

$$Z = \mathrm{tr} \exp(-\beta\mathcal{H}), \tag{6}$$

where $\beta = 1/k_B T$. In the high-temperature limit we can approximate Eq. (6) by

$$Z = \mathrm{tr}(1 - \beta\mathcal{H} + (\beta^2/2!)\mathcal{H}^2 - (\beta^3/3!)\mathcal{H}^3 \pm \cdots). \tag{7}$$

After performing the required traces we get

$$Z = 2^N \Big\{ 1 + N\frac{\beta^2}{2!}\left[\left(\frac{\gamma\hbar H}{2}\right)^2 + 3J_2^2 - 36J_2J_3 + 144J_3^2\right] - N\frac{\beta^3}{3!}\left[12\left(\frac{\gamma\hbar H}{2}\right)^2 (9J_3 - J_2) + 108J_3(J_2^2 - 12J_2J_3 + 36J_3^2)\right]$$
$$+ N\frac{\beta^4}{4!}\left[24\left(\frac{\gamma\hbar H}{2}\right)^2 (7J_2^2 - 132J_2J_3 + 600J_3^2) + \cdots\right] + N^2\frac{\beta^4}{4}\left(\frac{\gamma\hbar H}{2}\right)^2 (3J_2^2 - 36J_2J_3 + 144J_3^2) + \cdots \Big\}, \tag{8}$$

where N is the number of atoms in the solid. The pressure can be found directly by forming $\beta^{-1}\partial \ln Z/\partial V$. We get

$$P_V(T,H)=\frac{RJ_2}{Vk_B}\frac{d\ln|J_2|}{d\ln V}\left\{\frac{3J_3}{k_B}\left[[1-6a(1+b)+48a^2b]T^{-1}-\frac{6J_2}{k_B}a[2+b-12a(1+2b)+108a^2b]T^{-2}\right]\right.$$
$$\left.+2\left(\frac{\gamma\hbar H}{2k_B}\right)^2\left[(1-9ab)T^{-2}+\frac{J_2}{k_B}[7-66a(1+b)+600a^2b]T^{-3}+\cdots\right]+\cdots\right\}, \quad (9)$$

where

$$a=-J_3/J_2, \quad (10)$$

$$b=\frac{d\ln|J_3|/d\ln V}{d\ln|J_2|/d\ln V}; \quad (11)$$

V is the molar volume and R is the universal gas constant. The traces in Eqs. (8) and (9) were done for the specific case of bcc ^{3}He. The term due to $\mathrm{tr}[(\mathcal{H}_{ex}^{(2)})^3]$ is identically zero in bcc ^{3}He because there is no triple of atoms which are mutual near neighbors of one another. For convenience in comparing the relative sizes of the various contributions to the pressure, we write

$$\Delta P_V(T,H)=\Delta P_A+\Delta P_B+\Delta P_C+\Delta P_D, \quad (11a)$$

where

$$\Delta P_A\propto(J_2/k_B)^2[1-6a(1+b)+48a^2b]T^{-1}, \quad (11b)$$

$$\Delta P_B\propto(J_2/k_B)^3a[2+b-12a(1+2b)+108a^2b]T^{-2}, \quad (11c)$$

$$\Delta P_C\propto(J_2/k_B)H^2(1-9ab)T^{-2}, \quad (11d)$$

$$\Delta P_D\propto(J_2/k_B)^2H^2[7-66a(1+b)+600a^2b]T^{-3}. \quad (11e)$$

Table I lists these contributions at 40 and 60 kG for various values of a. Looking at Eq. (9) we see that the inclusion of triple exchange has two obvious effects on the pressure. First, it renormalizes the leading T^{-1} piece, thus changing the values of J_2 obtained by PA in zero field. Second, it softens the effect of the magnetic field, i.e., reduces the contribution from the T^{-2} piece. These two effects compete with one another, and the softening of the magnetic field piece must dominate if triple exchange is going to improve the agreement between theory and experiment.

TABLE I. Contributions to ΔP at $T^{-1}=35$ K^{-1}.

H (kG)	a	ΔP_A (10^{-4} atm)	ΔP_B (10^{-4} atm)	ΔP_C (10^{-4} atm)	ΔP_D (10^{-4} atm)
40	0	13.33	0.0	−7.11	1.19
40	$\frac{1}{30}$	13.33	0.38	−5.13	0.73
40	$\frac{1}{20}$	13.33	0.79	−3.53	0.40
60	0	13.33	0.0	−16.0	2.68
60	$\frac{1}{30}$	13.33	0.38	−11.6	1.65
60	$\frac{1}{20}$	13.33	0.79	−7.94	0.89

We now use a simple model of exchange, due to GZ, to estimate the values of a and b. GZ use an overlap integral between pairs and triples to estimate the respective exchange parameters. The overlap integral for a pair is a measure of the probability that atom 1 tunnels from lattice site 1 to lattice site 2 times the probability that atom 2 tunnels from lattice site 2 to lattice site 1. The overlap integral for triple exchange is defined analogously. The ratio of J_3 to J_2 is found to be

$$J_3/J_2=a\lesssim\exp[(-\tfrac{1}{2}\alpha^2)(\tfrac{2}{3}\Delta^2-\sigma^2)]\sim 0.1, \quad (12)$$

where $2\alpha^{-2}$ is the width of the Gaussian single-particle wave functions used to describe the solid, Δ is the near-neighbor distance, and σ is the diameter of the hard core associated with each ^{3}He atom. When a pair of atoms exchange, they must move far enough from the straight line connecting their lattice sites to avoid their respective hard cores. When three atoms cyclically permute they do not have to deviate very much from the most direct path connecting the three

lattice sites. Therefore, the parts of the wave functions which overlap are further out on the tail of the Gaussian for pairs than for triples. In obtaining Eq. (12) we ignored the hard cores in triple exchange which is why we write an inequality. We estimate the ratio of the overlaps for four particles and three particles to be

$$J_4/J_3 \cong \exp(-\alpha^2\Delta^2/6) \cong 0.04.$$

We also estimate the size of next-nearest-neighbor (nnn) exchange as compared to nearest-neighbor (nn) exchange and find that

$$J(\mathrm{nnn})/J(\mathrm{nn}) \cong \exp(-\alpha^2\Delta^2/6) \cong 0.04.$$

The detailed way in which nnn exchange enters the partition function will be examined in a subsequent paper. The volume dependence of exchange is principally due to the width of the single-particle wave functions used to describe the solid. Two tunneling atoms contribute four wave functions and three tunneling atoms contribute six wave functions. Hence, we would expect that the approximate volume dependence of J_3 is given by

$$d\ln|J_3|/d\ln V = \tfrac{3}{2} d\ln|J_2|/d\ln V. \quad (13)$$

The parameter a is given the sample values $\frac{1}{30}$, $\frac{1}{25}$, and $\frac{1}{20}$, and b is set equal to $\frac{3}{2}$. These values are substituted into Eq. (9) and compared with the experiment of KA at a molar volume of 23.88 cm^3 and magnetic fields of 40 and 60 kG. The results are shown in Fig. 1. We see that the discrepancy between theory and experiment in now much less drastic. The values of J_2 found by PA are increased by 30%–50% when triple exchange is included in the T^{-1} piece. We emphasize that the improved agreement between theory and experiment does not depend strongly on the particular values chosen for J_2, $d\ln|J_2|/d\ln V$, a, and b, but instead depends on the ferromagnetic nature of triple exchange.

The values of J_2 obtained by experiments inherently include the effect of near-neighbor exchange plus all higher-order tunneling motions. It has generally been assumed that these higher-order contributions would be too small to significantly affect the properties exhibited by the solid. The results of the experiment by Kirk and Adams plus the analysis presented here suggest that this assumption is not valid. The extent that triple exchange and higher terms contribute to the thermodynamics of solid ^{3}He will be the goal of future work.

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