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Nonlinear Frequency Shift of an Electron Plasma Wave*

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The shift in the complex frequency of a large-amplitude electron plasma wave is formulated in terms of a new subtraction procedure which reproduces the damping coefficient of Mazitov and of O'Neil directly from Poisson's equation. A time-dependent frequency shift is obtained which produces a phase shift that should be amenable to experimental observation. The results are interpretated in terms of simultaneous conservation of momentum and energy in the wave frame.

The calculations of Mazitov[1] and of O'Neil,[2] concerning the damping of a large-amplitude plasma wave, stimulated a series of interesting experiments on electron[3] as well as ion[4] waves, in which nonlinear oscillations of the wave amplitude were observed. In general, however, the nonlinearities give rise to oscillations in the phase of the wave as well as in its amplitude. In this paper we present a unified theory giving the time-dependent nonlinear shift in both the real and imaginary parts of the complex frequency of an electron plasma wave. The damping coefficient thus obtained agrees with that of Mazitov and that of O'Neil, while the corresponding shift in the real part of the frequency decreases from its initial zero value down to an asymptotic level about which it executes oscillations at twice the bounce frequency of electrons trapped in the potential well of the wave.

From an experimental point of view one is interested in the nonlinear shift in dispersion properties that occurs when the transmitter power is turned up and the receiver gain is turned down by a corresponding amount. Motivated by this physical consideration, we formulate a subtraction procedure which yields this nonlinear shift directly. Such a subtraction procedure is particularly useful in calculating the nonlinear shift in the real part of the frequency. The frequencies for both the linear and nonlinear cases are determined primarily by nonresonant electrons, whereas their difference (i.e., the nonlinear frequency shift) is determined essentially by resonant electrons. Consequently, it is important to use a calculational procedure that involves only those quantities that change between the linear and nonlinear solution.[5]

We note before proceeding with the mathematical formulation of the subtraction procedure that there are two physical situations that can be considered in this calculation. One case consists of an initial-value problem where the wave number is fixed and the frequency undergoes nonlinear oscillations as a function of time. However, experimentally the frequency is generally fixed and the wave number exhibits spatial oscillations. We carry out the calculation for the conceptually simpler case of an initial-value problem and provide later an analytic transformation between this case and the more realistic experimental situation. We start by expressing the single-mode electric field in the form

$$E(x,t)=E_0\exp\{i[kx-\int_0^t dt'\,\omega(t')]\}/2i+\text{c.c.} \quad (1)$$

which must satisfy Poisson's equation,

$$ikE_k(t)=-4\pi en\int_{-\infty}^{\infty}dv\int_{-\lambda/2}^{\lambda/2}(dx/\lambda)e^{-ikx}\,\delta f, \quad (2)$$

where δf is the exact nonlinear perturbation (i.e., the spatially inhomogeneous part of the distribution function). From both sides of this equation we subtract the linear charge density,

$$ikE_k(t)\epsilon(k,\omega(t))=-4\pi en\int_{-\infty}^{\infty}dv\,(\delta f_k-\delta f_k^{\,L}), \quad (3)$$

where

$$\delta f_k^L = \frac{e}{m}\int_0^t dt' E_k(t') \exp[ikv(t'-t)]\frac{\partial f_0}{\partial v} \quad (4)$$

is the solution to the Boltzmann equation, linearized about the initially homogeneous distribution f_0, and

$$\epsilon(k, \omega(t)) = 1 - \frac{(m/e)\omega_p^2}{ik}\int_{-\infty}^{\infty} dv \frac{\delta f_k^L}{E_k(t)} \quad (5)$$

is the linear dielectric coefficient evaluated at the instantaneous frequency.[6]

For small E_0 the propagation properties are determined by $\epsilon(k, \omega(t)) = 0$, which yields the linear solution [i.e., $\omega(t) = \omega_L$]. However, as E_0 is increased, the nonlinear correction to the charge density on the right-hand side (rhs) of Eq. (3) produces a small nonlinear shift $\delta\omega = \omega(t) - \omega_L$. To extract this shift, we perform an expansion of ϵ about ω_L in Eq. (3):

$$ik\left(\frac{\partial \epsilon}{\partial \omega}\right)_{\omega_L} \delta\omega E_k = -4\pi en\int_{-\infty}^{\infty} dv\, (\delta f_k - \delta f_k^L), \quad (6)$$

where the zero order term in the Taylor expansion [i.e., $\epsilon(k, \omega_L)$] vanishes by definition of ω_L. Relation (6) is free from the objectionable[7] boundary oscillations that may be introduced by arbitrarily dividing the velocity integration into a resonant and nonresonant region,[1,2] since here the subtraction procedure used allows the integrand on the rhs of Eq. (6) to find its own natural cutoff.

To evaluate the rhs of Eq. (6) we need to determine δf. Recalling that the distribution function is constant along particle trajectories allows us to express δf, for an initially homogeneous plasma, as $\delta f = f_0(v_0(x, v, t)) - f_0(v)$, where $v_0(x, v, t)$ is the initial velocity of a particle with coordinates (x, v) at time t. If the amplitude of the excitation is not made too large, the region where δf_k differs significantly from δf_k^L is confined to the neighborhood of the linear phase velocity v_p. Therefore, the velocity integral of Eq. (6) has its main contribution from this region, and we may form a Taylor expansion of δf around v_p. Quantitatively, this expansion requires that the field amplitude be small enough that $(eE_0/mk)^{1/2} \ll v_T^2/v_p$, where v_T is the electron thermal velocity.[8] In terms of the velocity perturbation $\Delta v \equiv v_0(x, v, t) - v$, the Taylor expansion can be written as

$$\delta f = \left(\frac{\partial f_0}{\partial v}\right)_{v_p} \Delta v + \frac{1}{2}\left(\frac{\partial^2 f_0}{\partial v^2}\right)_{v_p} [(\Delta v)^2 + 2(v - v_p)\Delta v]. \quad (7)$$

The orbits in this resonant region can be calculated by neglecting the amplitude and phase changes of the wave, provided $\gamma_L \ll (eE_0 k/m)^{1/2} \equiv \omega_B$, where γ_L is the Landau damping coefficient.[1,2] The latter inequality is in the opposite direction from that used in obtaining Eq. (7). Therefore, their combination determines the range of validity of our procedure; it works only for waves with phase velocities such that $v_p \gtrsim 4v_T$.[8] Integration of the equations of motion, under the above assumption, yields Δv as a Fourier series of the scaled time variable $\omega_B t$ with its coefficients being combinations of elliptic integrals.[1,2] With this Δv, Eq. (7) gives a δf that permits the evaluation of the rhs of Eq. (6).

The shift in the complex frequency can then be consistently extracted to first order in γ_L/ω_B by neglecting the damping of E_k in Eq. (6).[9] By equating imaginary parts the damping coefficient shift $\delta\Gamma$ is found. This damping shift, displayed in Fig. 1, corresponds identically to that exhibited in Eq. (31) of Ref. 2. Likewise, equating real parts in Eq. (6) yields the frequency shift $\delta\Omega$, whose time evolution is shown in Fig. 2. The time-dependent expression for $\delta\Omega$, consisting of several double summations of integrations over the modulus of the elliptic integrals, is very cumbersome and is not written out here, how-

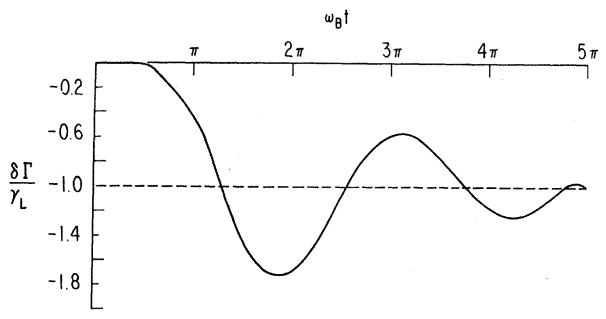

FIG. 1. Damping coefficient shift versus time.

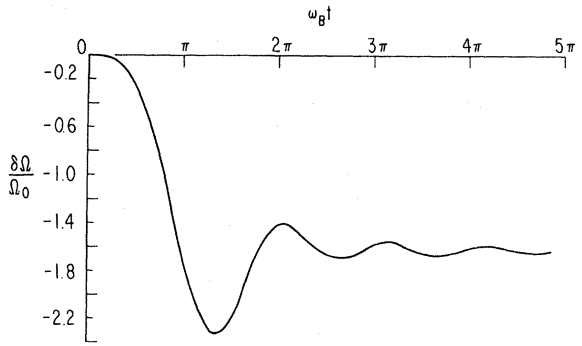

FIG. 2. Frequency shift versus time.

ever; we do show its simple asymptotic form:

$$\delta\Omega(\infty) = -\Omega_0 \frac{16}{\pi}\int_0^1 dK\left[\left(\frac{K}{F}\right)(2E-F)^2 + \frac{[2(E-F)+K^2F]^2}{K^6F}\right],$$
$$\Omega_0 \equiv \left(\frac{eE_0}{mk}\right)^{1/2}\left(\frac{\omega_p}{k}\right)^2\left(\frac{\partial^2 f_0}{\partial v^2}\right)_{v_p}\left(\frac{\partial\epsilon}{\partial\omega}\right)_{\omega_L}^{-1}, \tag{8}$$

where F and E are the complete elliptic integrals of the first and second kind, respectively, and K their modulus. The first term in the integrand of Eq. (8) gives the contribution of trapped particles, $-1.40\Omega_0$, while the second is due to the free but nearly resonant ones, whose contribution is $-0.23\Omega_0$. The numerical value of the time-asymptotic frequency shift given by Eq. (8) is remarkably close to the constant shift that Manheimer and Flynn[10] find in connection with the formation of time-independent Bhatnagar-Gross-Krook modes.

The physical basis of the shifts in the damping coefficient and in the frequency of a large-amplitude plasma wave can be understood through two conservation relations. We derive them from Poisson's equation and the collisionless Boltzmann equation. The first states that momentum is conserved between the resonant particles, as defined naturally by the subtraction procedure, and the background electrons which support the wave:

$$\frac{d}{dt}\left[n\int_{-\infty}^{\infty}dv\int_{-\lambda/2}^{\lambda/2}\frac{dx}{\lambda}\,mv(\delta f-\delta f_{\mathrm{QL}})\right] = -\delta\Gamma\, k\left(\frac{\partial\epsilon}{\partial\omega}\right)_{\omega_L}\frac{|E_k|^2}{2\pi}, \tag{9}$$

where δf_{QL} is defined to satisfy the quasilinearlike equation

$$\frac{\partial}{\partial t}\delta f_{\mathrm{QL}} \equiv \frac{\partial}{\partial v}\left(\sum_k \frac{e}{m}E_k\delta f_{-k}^{\,L}\right). \tag{10}$$

The second relation,[11]

$$\frac{d}{dt}\left[n\int_{-\infty}^{\infty}dv\int_{-\lambda/2}^{\lambda/2}\frac{dx}{\lambda}\,\frac{mv^2}{2}(\delta f-\delta f_{\mathrm{QL}})\right] = \left(2\gamma_L - \frac{d}{dt}\right)\left[\left(\frac{\partial\epsilon}{\partial\omega}\right)_{\omega_L}\delta\Omega\,\frac{|E_k|^2}{2\pi}\right], \tag{11}$$

states that (to lowest order in γ_L/ω_B) a change in the kinetic energy of the resonant particles, as seen in the wave frame, must be balanced by a corresponding change in the total wave energy $[\partial(\epsilon\omega)/\partial\omega]E_k^2/2\pi$, which in this frame of reference is directly proportional to the nonlinear frequency shift. The unified picture that emerges from these two conservation equations consists, therefore, of a resonant wave-particle interaction in which the amplitude of the wave oscillates in order to conserve momentum, while the frequency decreases in such a manner that energy is conserved.

The use of a simple phase-space diagram for a trapped particle, such as Fig. 3, is helpful in visualizing the relevance of the above conservation relations to our quantitative results. According to the approximation of Eq. (7), the total momentum change experienced by the four conjugate test points [(1, 3) and (2, 4) of Fig. 3] after a half-cycle rotation in phase space is $-4mv_0^2(\partial f_0/\partial v)v_p$, i.e., it is independent of the second derivative. This explains why we do not find a correction to the damping coefficient of Mazitov and of O'Neil, although we work to higher order. On the other hand, the change in the total kinetic energy of the same group of test points, after a quarter cycle, is given by $-mv_0^4(\partial^2 f_0/\partial v^2)v_p$, which does not contain the first derivative. That is the physical reason why one must work to higher order in the resonance region in order to extract a frequency shift. A further qualitative point that can be seen from this phase-space picture concerns the time dependence of $\delta\Gamma$ and $\delta\Omega$. Momentum, being an odd function of velocity, is regenerated (for a given particle energy) after a complete

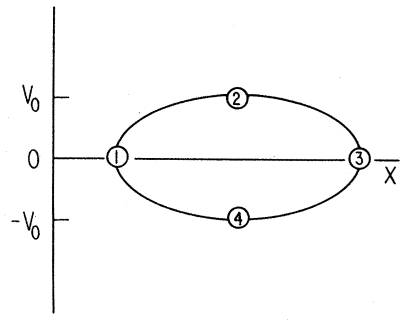

FIG. 3. Phase-space trajectory of a trapped particle.

rotation in phase space. Therefore, the oscillations in $\delta\Gamma$, Fig. 1, occur at a frequency near ω_B. The oscillations in $\delta\Omega$, Fig. 2, take place at essentially twice this rate because they reflect changes in kinetic energy, which is an even function of velocity.

We have also solved the problem for the experimentally more realistic case where the frequency is fixed and the wave number exhibits nonlinear oscillations. For this case, the nonlinear shift in wave number is given by $\delta k = \delta\Omega(k_B x)/(-v_g)$, where v_g is the wave group velocity and $k_B \equiv \omega_B/v_p$ is the bounce wave number. The functional dependence of $\delta\Omega(k_B x)$ is given in Fig. (2), with $k_B x$ replacing $\omega_B t$. To illustrate the feasibility of measuring such a wave-number shift, we note that for typical experimental parameters[3] ($v_p/v_g = 2.5$, $v_p/v_T \simeq 4$, $\omega_B/kv_p = 0.08$) one obtains an asymptotic shift of 7 rad/m. The spatial oscillation in phase has an amplitude of about 1 rad. These quantities should be measurable.

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[11]The operator $(2\gamma_L - d/dt)$ appears on the right-hand side of Eq. (11) because our formulation is in terms of $\delta\Gamma$, rather than the full damping coefficient.

[12]The "radiation pressure" calculated by W. E. Drummond [Phys. Fluids 7, 816 (1964)] produces a change in the plasma density that also gives rise to a wave-number shift. We find this to be of order $\delta k/k \sim (\omega_B/\omega)^4(v_p/v_T)^2$, which is smaller than the effect we calculate. Also, in a real experiment one must make sure that the large-amplitude wave does not change the density by such indirect methods as modification of boundary conditions or injection regions.

Magnetic Pressure of Solid ^{3}He

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The pressure due to the magnetic ordering in solid ^{3}He is calculated using a Hamiltonian which includes triple exchange. This added degree of freedom supplies a simple explanation for the observed behavior of the pressure as a function of temperature.

The measurement of the pressure change as a function of temperature in solid ^{3}He was first used by Panczyk and Adams[1] (PA) to obtain a value for J_2, the pair-exchange frequency. Three groups of experimentalists[2-4] have recently repeated these measurements in the presence of a large external magnetic field. The results of the three experiments disagree quantitatively from the result expected on assuming solid ^{3}He to be a "good" spin-$\frac{1}{2}$ Heisenberg solid.[5] The Heisenberg Hamiltonian includes only the possibility of pair interactions, which for ^{3}He are antiferromagnetic and on the order of 1 mK. Triple exchange has been shown by Thouless[6] to be inherently ferromagnetic. The relative magnitude of triple exchange compared with pair exchange has been estimated by Guyer and Zane[7] (GZ). This paper will include triple exchange in the Hamiltonian used to calculate the pressure due to spin ordering. The results we obtain are substantially in agreement with the experiment of Kirk and Adams[3] (KA).

The usual Hamiltonian written for solid ^{3}He in an external magnetic field is

$$\mathcal{H} = \mathcal{H}_{ex}^{(2)} + \mathcal{H}_Z, \tag{1}$$