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[10]It is possible that higher-order terms in p may give significant contributions to the dispersion relation for phonons thermally excited at 0.35°K. Also it has been proposed [A. Molinari and T. Regge, Phys. Rev. Lett. 25, 1531 (1971)] that for small p the dispersion relation may be of the more general form $\epsilon(p)=c_0p(1-\gamma_1 p-\gamma_2 p^2+\cdots)$ with γ_1 negative. These questions will presumably be resolved when specific heat measurements at lower temperatures are made.

[11]Phillips, Waterfield, and Hoffer (Ref. 4) analyzed their specific heat data treating both γ and c_0 as adjustable parameters. They obtained a best fit with $\gamma=-4.1\times10^{37}$ g^{-2} cm^2 sec^2 and $c_0=2.397\times10^4$ cm sec^{-1}. This value of c_0 is considerably outside the experimental uncertainty of the independent measurement of c_0 made in Ref. 9. Consequently we adopted the alternative analysis in which only γ was determined by the specific heat results.

Temporal Behavior of Electron Distributions in an Electric Field

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Coefficients are derived for a one-dimensional Fokker-Planck equation describing the evolution of an electron energy distribution. These coefficients include acceleration of electrons between collisions in addition to the collision terms. The coefficient for the average rate of change of the electron energy, $\langle\Delta\epsilon\rangle/\Delta t$, is the same as obtained with the "average-electron" theory; but the expression for the dispersion, $\langle(\Delta\epsilon)^2\rangle/\Delta t$, has not appeared previously.

The computation of electron energy distributions in an electric field has numerous applications, such as the avalanche breakdown of an insulating gas in a waveguide and the breakdown caused by a laser beam focused on an optical crystal.[1] Traditionally this problem has been treated by a Legendre expansion of the distribution function in three-dimensional velocity space.[2-4] However, practical considerations require truncating the expansion after the first terms, and mathematical complications have limited this method to essentially time-independent problems. The objective of this work is a derivation of the coefficients of a time-dependent Fokker-Planck operator, differing from previous Fokker-Planck operators by including the acceleration of electrons between collisions.

Let $f(\epsilon,t)d\epsilon$ be the expected number of electrons with energies between ϵ and $\epsilon+d\epsilon$. For such a one-dimensional function, it is shown[5] that

$$\frac{\partial f}{\partial t}=-\frac{\partial}{\partial\epsilon}\left(\frac{\langle\Delta\epsilon\rangle}{\Delta t}f\right)+\frac{1}{2}\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\langle(\Delta\epsilon)^2\rangle}{\Delta t}f\right), \tag{1}$$

where

$$\langle(\Delta\epsilon)^n\rangle=\int P(\epsilon|\Delta\epsilon,\Delta t)(\Delta\epsilon)^n d(\Delta\epsilon), \tag{2}$$

with $P(\epsilon|\Delta\epsilon,\Delta t)d(\Delta\epsilon)$ equal to the probability that the electron energy will change from ϵ to the range $d(\Delta\epsilon)$ about $\epsilon+\Delta\epsilon$ during the time Δt.

The basic assumptions made in deriving Eq. (1) are that a time increment Δt can be chosen which is long compared to the time between collisions but short compared to the time in which a significant change occurs in the electron energy, and also that $\langle(\Delta\epsilon)^n\rangle$ is proportional to Δt for $n=1$ or 2, but is proportional to Δt^2 or higher orders for n greater than 2.

Our derivation of the Fokker-Planck coefficients starts with the definition of the probability $P_n(t_1,\cdots,t_n)dt_1\cdots dt_n$ that the electron will have exactly n collisions between t and $t+\Delta t$, and that the first will occur between t_1 and t_1+dt_1, the second between t_2 and t_2+dt_2, etc. If ν is the effective collision frequency, then $P_n(t_1,\cdots,t_n)=\nu^n e^{-\nu\Delta t}$, where the t_j are ordered times.

Now let ϵ_j be the energy immediately before the jth collision and let $Y_j\epsilon_j$ and θ_j be the energy and the angle between the velocity vector and the electric field after that collision. Computing the acceleration of the electron, we have the following deterministic relation between ϵ_{j+1} and ϵ_j:

$$\epsilon_{j+1} = Y_j\epsilon_j + (e^2/2m)\psi^2(t_j, t_{j+1}) + e(2\epsilon_j Y_j/m)^{1/2}\cos(\theta_j)\psi(t_j, t_{j+1}), \tag{3}$$

where

$$\psi(t_j, t_{j+1}) = \int_{t_j}^{t_{j+1}} E(t)\,dt.$$

By mathematical induction,

$$\epsilon_{j+1} = \epsilon_0 \prod_{k=1}^{j} Y_k + \frac{e^2}{2m}\sum_{k=1}^{j+1}\psi^2(t_{k-1}, t_k)\prod_{l=k}^{j} Y_l + e\sum_{k=1}^{j+1}\left(\frac{2Y_{k-1}\epsilon_{k-1}}{m}\right)^{1/2}\cos(\theta_{k-1})\psi(t_{k-1}, t_k)\prod_{l=k}^{j} Y_l, \tag{4}$$

where e_0 is the energy at the beginning of increment Δt and, if there are n collisions, ϵ_{n+1} is defined as the energy at the end of Δt. Since the change in energy is small in time Δt, we take ϵ_{k-1} in the square root in Eq. (4) equal to ϵ_0.

In general Y_k will be correlated with θ_k. Here, however, we make the assumption (reasonable for electrons colliding with heavy molecules) that there is no energy interchange ($Y_j = 1$) and θ_j is isotropically distributed in the laboratory system. The energy change when n collisions occur in Δt is

$$\epsilon_{n+1} - \epsilon_0 = \frac{e^2}{2m}\sum_{k=1}^{n+1}\psi^2(t_{k-1}, t_k) + e\sum_{k=1}^{n+1}\left(\frac{2\epsilon_0}{m}\right)^{1/2}\cos(\theta_{k-1})\psi(t_{k-1}, t_k) \tag{5}$$

and the expectation value when n isotropic collisions occur at the specified times t_j is

$$\langle(\epsilon_{n+1} - \epsilon_0)\rangle = \frac{e^2}{2m}\sum_{k=1}^{n+1}\psi^2(t_{k-1}, t_k), \tag{6}$$

and

$$\langle(\epsilon_{n+1} - \epsilon_0)^2\rangle = \frac{e^4}{4m^2}\sum_{k=1}^{n+1}\sum_{l=1}^{n+1}\psi^2(t_{k-1}, t_k)\psi^2(t_{l-1}, t_l) + \frac{2e^2}{3}\frac{\epsilon_0}{m}\sum_{k=1}^{n+1}\psi^2(t_{k-1}, t_k). \tag{7}$$

Then, the expression for the Fokker-Planck coefficients is

$$\langle(\Delta\epsilon)^n\rangle = \sum_{n=0}^{\infty}\int_0^{\Delta t} dt_n \cdots \int_0^{t_2} P_n(t_1, \cdots, t_n)\langle(\epsilon_{n+1} - \epsilon_0)^n\rangle\, dt_1. \tag{8}$$

Now assume that the electric field $E(t)$ is a dc field, essentially unchanged during time Δt. Then $\psi(t_{k-1}, t_k) = E(t)(t_k - t_{k-1})$, and from Eqs. (6) and (8), we obtain

$$\langle\Delta\epsilon\rangle = \frac{e^2E^2(t)}{m\nu^2}e^{-\nu\Delta t}\sum_{n=0}^{\infty}\frac{n+1}{(n+2)!}(\nu\Delta t)^{n+2} = \frac{e^2E^2}{m\nu^2}(\nu\Delta t - 1 + e^{-\nu\Delta t}) \cong \frac{e^2E^2}{m\nu}\Delta t, \tag{9}$$

where the last step follows because we consider time increments such that $\nu\Delta t \gg 1$. Similarly, from Eqs. (7) and (8), we obtain

$$\begin{aligned}\langle(\Delta\epsilon)^2\rangle &= \left(\frac{e^2E^2}{m\nu^2}\right)^2 e^{-\nu\Delta t}\sum_{n=0}^{\infty}\frac{(n+6)(n+1)}{(n+4)!}(\nu\Delta t)^{n+4} + \frac{4}{3}\frac{e^2\epsilon}{m\nu^2}E^2 e^{-\nu\Delta t}\sum_{n=0}^{\infty}\frac{n+1}{(n+2)!}(\nu\Delta t)^{n+2} \\ &= \left(\frac{e^2E^2}{m\nu^2}\right)^2\{[(\nu\Delta t)^2 - 6] + 2[(\nu\Delta t)^2 + 3\nu\Delta t + 3]e^{-\nu\Delta t}\} + \frac{4}{3}\frac{e^2\epsilon}{m\nu^2}E^2[(\nu\Delta t - 1) + e^{-\nu\Delta t}] \\ &= \left(\frac{e^2E^2}{m\nu}\right)^2(\Delta t)^2 + \frac{4}{3}\frac{e^2E^2}{m\nu}\epsilon\Delta t,\end{aligned} \tag{10}$$

where, in the last step, we again take $\nu\Delta t \gg 1$. Therefore, neglecting terms of $(\Delta t)^2$ or higher, we have

$$\frac{\langle(\Delta\epsilon)^2\rangle}{\Delta t} = \frac{4}{3}\frac{e^2E^2}{m\nu}\epsilon. \tag{11}$$

Now, consider the opposite limiting case in which $E = E_0\sin\omega t$ and $\omega \gg \nu$, the usual situation for laser radiation. Then

$$\psi(t_j, t_{j+1}) = E_0(\cos\omega t_j - \cos\omega t_{j+1})/\omega,$$

and, since the time between t_j and t_{j+1} is much greater than ω^{-1}, we can take average values for ψ^2 in Eq. (8). Then, from Eqs. (6) and (8), we obtain, in the appropriate limit,

$$\langle\Delta\epsilon\rangle/\Delta t = e^2 E_0^2 \nu/2m\omega^2. \tag{12}$$

Similarly, from Eqs. (7) and (8), after using the relationship

$$(\omega/E_0)^4 \int_0^{\Delta t} dt_n \cdots \int_0^{t_2} \psi^2(t_j, t_{j+1})\psi^2(t_k, t_{k+1}) P_n dt_1 = \begin{cases} \frac{9}{4}p_n, & \text{for } j = k, \\ \frac{9}{8}p_n, & \text{for } j = k \pm 1, \\ p_n, & \text{otherwise}, \end{cases}$$

where p_n is a Poisson distribution, we obtain

$$\begin{aligned} \langle(\Delta\epsilon)^2\rangle &= \left(\frac{e^2 E_0^2}{2m\omega^2}\right)^2 e^{-\nu\Delta t} \sum_{n=0}^{\infty} \frac{(n^2 + \frac{7}{2}n + \frac{9}{4})}{n!}(\nu\Delta t)^n + \frac{2}{3}\frac{\epsilon}{m}\frac{e^2 E_0^2}{\omega^2} e^{-\nu\Delta t} \sum_{n=0}^{\infty} \frac{(n+1)}{n!}(\nu\Delta t)^n \\ &= \left(\frac{e^2 E_0^2}{2m\omega^2}\right)^2 [(\nu\Delta t)^2 + \tfrac{9}{2}\nu\Delta t + \tfrac{9}{4}] + \frac{2}{3}\frac{\epsilon}{m}\frac{e^2 E_0^2}{\omega^2}(\nu\Delta t + 1) \\ &\cong \frac{2}{3}\frac{\epsilon}{m}\frac{e^2 E_0^2}{\omega^2}\nu\Delta t. \end{aligned} \tag{13}$$

The result is that the Fokker-Planck equation can be generally written as

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial\epsilon}(Kf) + \frac{2}{3}\frac{\partial^2}{\partial\epsilon^2}(\epsilon K f), \tag{14}$$

where $K = \langle\Delta\epsilon\rangle/\Delta t$ is given for slowly varying electric fields by Eq. (9), and in the opposite limit by Eq. (12); where K through ν is generally a function of energy.

The validity of this equation requires Δt to be chosen large compared to the collision time ν^{-1}; yet small compared to the time for the electron to gain the characteristic energy ϵ_i. This will always be possible if the energy gained in one collision time is small compared to ϵ_i, i.e., if

$$e^2E^2/m\nu^2\epsilon_i \ll 1. \tag{15}$$

The relation in Eq. (22) is also required for our implicit assumption that θ_0 is isotropically distributed.

A sample application of Eq. (14) is given in Figs. 1 and 2 which show the evolution of a distribution function which initially was a narrow Gaussian centered at 10 eV. The calculation involved a finite-difference computer program and invoked

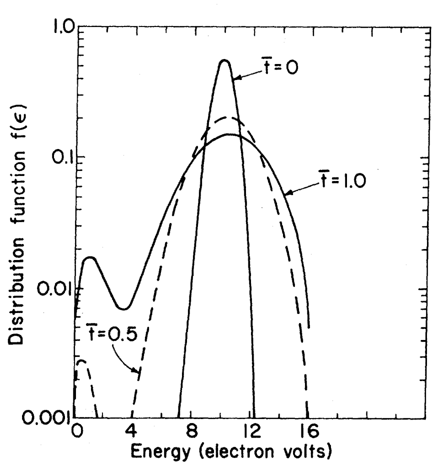

FIG. 1. The evolution of an electron distribution function initially a Gaussian $\pi^{-1/2}\exp[-(\epsilon-10)^2]$. As time goes on, the Gaussian spreads, each electron reaching the ionization energy produces 1.2 electrons at the bottom of the energy scale, and a secondary hump appears at low energies.

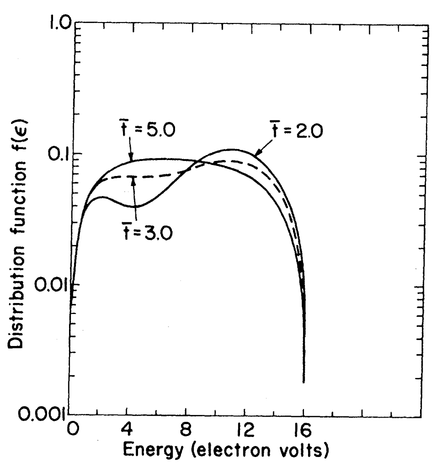

FIG. 2. Further evolution of the distribution function in Fig. 1. By $\bar{t}=5$, a time considerably shorter than the e-folding time, it has attained a nearly constant shape such that $f(\epsilon,t) = e^{t/\tau}f(\epsilon)$.

the boundary conditions $f(\epsilon_i)=0$, $(\epsilon_i=16$ eV), and that each electron attaining the ionization energy ϵ_i disappears and is replaced by $2p_a$ electrons at zero energy. This is appropriate for SF_6, in which attachment occurs mainly in a narrow band[6] below about 0.1 eV, where p_a is the probability of escaping the attachment band, $(2\alpha-\beta)/2\alpha$. For the numerical example $p_a=0.6$ which, based on Pederson's empirical equation,[7] corresponds to $E/p=126.5$ V/cm Torr. We use the low-frequency Eq. (9) for K, and take ν proportional to ϵ (a relationship suggested by matching the SF_6 radius to the sum of theoretically calculated S and F radii[8]). The figures are plotted with energy in eV and time measured in units of $1/K_1$, where K_1 is the value of K at 5 eV. Numerically, $1/K_1$ is 4.1×10^{-13} sec and $\nu=5.4\times10^{12}\epsilon$ when p is 1520 Torr of SF_6 and $E/p=126.5$ V/cm Torr with ϵ in eV.

Although our derivation proceeded independently from the Boltzmann equation, Eq. (14) is consistent with Allis's equation[3] for $f_0^{\,0}(v)$ (where $f_0^{\,0}$ is the first term in the Legendre expansion) provided we recall that $f(\epsilon)$ is proportional to $vf_0^{\,0}(v)$, that our derivation assumes $Y_j=1$, and that the Legendre methods assume $f_0^{\,0}$ independent of time. Although no precise criterion for truncating the Legendre expansion has been given,[3] if Eq. (15) were violated the distribution function would be highly anisotropic. Including time dependence allows us to treat arbitrary initial energy distributions and rapidly exponentiating avalanches. For cold background gases,[9] we can remove the restriction to zero energy loss by averaging Y_j in Eq. (4) to obtain $\overline{Y}_j=1-2m/M$.

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Light Scattering from Density Variations Excited in He^4 by a Thermal Transducer*

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We have observed light scattering from the density variations in the first- and second-sound modes generated simultaneously by a thermal transducer in superfluid He^4. Measurements of the temperature dependence of the density variations in both sound waves agree with the theoretical treatment by Lifshitz of the density-temperature coupling.

Recent reports[1] of measurements of the pressure variations in the sound waves generated by a thermal transducer in He^4 are in conflict with the theory of Lifshitz.[2] In this Letter, we report direct measurements of the density variation in such sound waves by light-scattering techniques. These measurements are found to agree with the Lifshitz theory.

The hydrodynamic equations for superfluids predict that two wave modes, each with its own characteristic velocity, are generated by a periodically heated, stationary plane surface (thermal transducer). Most of the energy is in the second-sound mode (velocity u_2) which, although it is predominantly a temperature or entropy wave, carries some density variations (ρ_2') as well. In addition to second sound, a low-intensity first-sound wave (velocity u_1) is simultaneously generated, which also carries density variations (ρ_1'). In pure He^4, the coupling between temperature and density variations is due primarily to the small, but finite, thermal expansion coefficient.

There are therefore two density-wave fields present in the superfluid. Since light is scattered by the periodic spatial modulation of the dielec-