tribution, one-dimensional, axially magnetized plasma column.

[9]H. Ikezi and R. J. Taylor, Phys. Rev. Lett. 22, 923 (1969), and J. Appl. Phys. 41, 738 (1970); R. J. Taylor, D. R. Baker, and H. Ikezi, Phys. Rev. Lett. 24, 206 (1970).

[10]K. Takayama and H. Ikegami, Phys. Rev. Lett. 5, 238 (1960).

[11]It is worth noting that the sign of $\vec{k}_2$ is reversed in the conversion with $\omega_0 > \omega_2$.

[12]In using the rf probe in a plasma, the grids are surrounded by sheaths. In an equivalent probe circuit the sheaths can be approximated by capacitors $C_{eq} \approx \epsilon_0 A/3\lambda_D$ in series with a voltage source. Since the load resistance $Z_{res} \approx 100$ kΩ is much larger than the sheath reactance $2/\omega C_{eq} \approx 3.5$ kΩ, the error in the measured electric field is small.

[13]A. Y. Wong, M. V. Goldman, F. Hai, and R. Rowberg, Phys. Rev. Lett. 21, 518 (1968).

[14]R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory*, edited by T. M. O'Neil and D. L. Book (Benjamin, New York, 1969).

[15]D. F. DuBois and M. V. Goldman, Phys. Rev. Lett. 28, 218 (1972).

Attenuation and Velocity of Sound in Superfluid Helium*

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The attenuation and velocity of sound in liquid He^4 have been calculated using the Landau-Khalatnikov kinetic equations and the phonon Boltzmann equation. A detailed comparison between theory and experiment is made at 0.35°K and good agreement is obtained over a wide range of requencies.

In this Letter we consider the attenuation and velocity of sound in liquid He^4. We will concentrate on the temperature range below 0.6°K where rotons may be neglected. Although there has been much theoretical effort, no satisfactory explanation of the attenuation and velocity at these temperatures has yet been given.[1] The theories have generally assumed that the energy-momentum relation for low-energy phonons has the form

$$\epsilon(p) = c_0 p(1 - \gamma p^2 + \cdots), \tag{1}$$

where c_0 and γ are positive quantities. In a previous Letter[2] it was proposed that γ is negative, thus making the dispersion anomalous in that the group velocity v_p increases with increasing p in the small-p regime. This idea radically changes the traditional approach[3] to phonon-phonon interactions in He^4 because three-phonon collisions that conserve energy and momentum may now occur. The proposal that $\gamma < 0$ has since received support from specific-heat measurements.[4]

In this Letter we report the results of detailed calculations of the attenuation and velocity of sound assuming that $\gamma < 0$. The agreement between these calculations and the experimental results of Abraham *et al.*[1] and Waters, Watmough, and Wilks[5] is remarkably good and constitutes strong evidence that γ is indeed negative. We also propose additional experiments to test the theory.

The starting point of our calculation is the kinetic equations of Landau and Khalatnikov.[3] These are

$$\partial\rho/\partial t + \mathrm{div}(\rho\vec{v}_s + \int \vec{p} n_p \, d\tau_p) = 0, \tag{2}$$

$$\partial\vec{v}_s/\partial t + \nabla[\mu_0 + \tfrac{1}{2} v_s^2 + \int(\partial\epsilon/\partial\rho) n_p \, d\tau_p] = 0, \tag{3}$$

where ρ is the density, $\vec{v}_s$ the superfluid velocity, μ_0 the chemical potential at absolute zero, and n_p the number of phonons of momentum $\vec{p}$. The integrals are over all of momentum space. We look for a solution of these equations in the form of a wave propagating in the z direction with wave vector $\vec{K}$ and frequency Ω. Consequently, we define $\Delta\rho$ and Δn_p by

$$\rho = \rho_0 + \Delta\rho \exp[i(2\pi K z - \Omega t)], \tag{4}$$

$$n_p = \tilde{n}_p + \Delta n_p \exp[i(2\pi K z - \Omega t)], \tag{5}$$

where

$$\tilde{n}_p = \{\exp[\beta(\epsilon + \vec{p}\cdot\vec{v}_s)] - 1\}^{-1}, \tag{6}$$

ρ_0 is the density at $T = 0$°K, and $\beta = (k_B T)^{-1}$. Note that $\tilde{n}_p$ depends upon z and t because $\vec{v}_s$ is space and time dependent, and also because ϵ depends on ρ. The attenuation α and the velocity correction Δc relative to the velocity c_0 at absolute zero

can then be shown to be

$$\alpha = -\frac{\rho_0 \Omega}{2c_0^3}\int\left(\frac{\partial\epsilon}{\partial\rho}+\frac{pc_0\cos\theta}{\rho}\right)\mathrm{Im}\left(\frac{\Delta n_p}{\Delta\rho}\right)d\tau_p, \tag{7}$$

$$\frac{\Delta c}{c_0}=\frac{\rho_0}{2c_0^2}\int\left(\frac{\partial\epsilon}{\partial\rho}+\frac{pc_0\cos\theta}{\rho}\right)\mathrm{Re}\left(\frac{\Delta n_p}{\Delta\rho}\right)d\tau_p+\frac{\rho_0}{2c_0^2}\int\frac{\partial^2\epsilon}{\partial\rho^2}n_p{}^0 d\tau_p-\frac{\beta\rho_0}{2c_0^2}\int\left(\frac{\partial\epsilon}{\partial\rho}+\frac{pc_0\cos\theta}{\rho}\right)^2 n_p{}^0(n_p{}^0+1)d\tau_p. \tag{8}$$

To calculate Δn_p we use the phonon Boltzmann equation[3]:

$$\frac{\partial n_p}{\partial t}=\left(\frac{\partial n_p}{\partial t}\right)_{\mathrm{coll}}+\frac{\partial n_p}{\partial\vec{p}}\cdot\frac{\partial H_p}{\partial\vec{r}}-\frac{\partial n_p}{\partial\vec{r}}\cdot\frac{\partial H_p}{\partial\vec{p}}, \tag{9}$$

where $H_p=\epsilon+\vec{p}\cdot\vec{v}_s$. The first term on the right-hand side is the rate of change of n_p due to phonon-phonon collisions. Since we are considering the propagation of a small-amplitude disturbance, we may linearize this equation. Using Eqs. (4) and (5) gives

$$\Delta n_p\left(1-\frac{v_p\cos\theta}{c_0}\right)-\beta n_p{}^0(n_p{}^0+1)\Delta\rho\left(\frac{\partial\epsilon}{\partial\rho}+\frac{pc_0\cos\theta}{\rho}\right)+\frac{i}{\Omega}\int C(\vec{p},\vec{p}')\Delta n_{p'}\,d\tau_{p'}=0. \tag{10}$$

Phonon-phonon collisions are now represented by the last term in this equation. Previous calculations[6,7] have used simplified forms for the kernel $C(\vec{p},\vec{p}')$ of the collision integral, these forms usually being equivalent to some variation of a relaxation-time approximation. However, here we have not made any approximation of this sort and instead have calculated $C(\vec{p},\vec{p}')$ by considering the details of three-phonon collisions. The resulting expression is complicated and will be given in a later paper. We have treated the phonon collisions in such detail because γp^2 in Eq. (1) is small (typically $\sim 10^{-2}$) and consequently the collisions between phonons are all small angle. Thus the way in which the thermal phonons come to equilibrium is not likely to be well described by a relaxation-time approximation.

We have solved Eq. (10) for Δn_p numerically by iteration and have then found α and $\Delta c/c_0$ from Eqs. (7) and (8). We used the experimentally determined values[8] for

$$\frac{\rho}{\epsilon}\frac{\partial\epsilon}{\partial\rho}=2.84, \tag{11}$$

$$\frac{\rho^2}{\epsilon}\frac{\partial^2\epsilon}{\partial\rho^2}=0.19. \tag{12}$$

These results are only valid for small p, since they assume that ϵ depends on ρ only through the dependence of c_0 on ρ. At larger values of p there will be significant corrections because of the variation of γ with density[4] [see Eq. (1)]. This *decreases* $\partial\epsilon/\partial\rho$ below the value given by Eq. (11). We estimate that because of this effect our theoretical results for α at 0.35°K may be too high by as much as 20%. The error in our calculation of $\Delta c/c_0$ is harder to estimate because of considerable cancelation between the different terms in Eq. (8). At 0.35°K a reasonable guess at the uncertainty in $\Delta c/c_0$ would be $\pm 20\times 10^{-6}$.

The velocity of sound c_0 has been very carefully determined by Whitney and Chase.[9] Their result is

$$c_0=(2.383\pm 0.001)\times 10^4\ \mathrm{cm\ sec^{-1}}. \tag{13}$$

If we assume that the energy-momentum relation[10] is correctly given by Eq. (1), then specific heat measurements enable γ to be determined. Taking the specific heat results of Phillips, Waterfield, and Hoffer[4] at 0.35°K gives[11]

$$\gamma=-8\times 10^{37}\ \mathrm{g^{-2}\ cm^{-2}\ sec^2}. \tag{14}$$

The integral over momentum space in Eq. (10) was performed using a mesh consisting of five values of the magnitude of $\vec{p}$ and 23 or 29 values of θ. The θ mesh had a variable spacing to provide more points in the region near $\theta=0$ where Δn_p varies rapidly with θ. We estimate that errors in α due to the use of a finite mesh of points are less than $\pm 20\%$. The error in $\Delta c/c_0$ may be larger than this because of cancelation between the three terms in Eq. (8). Various iteration schemes were used at different frequencies and temperatures. In some cases as many as 150 iterations were needed to obtain satisfactory convergence.

The experimental data for the velocity and attenuation have been discussed in great detail by Abraham *et al.*[1] For brevity we concentrate on two features of the data which have found no satisfactory explanation in terms of previous theories: (a) The velocity of sound in the temperature range 0.3 to 0.5°K is found to decrease with increasing frequency between 12 and 84 MHz

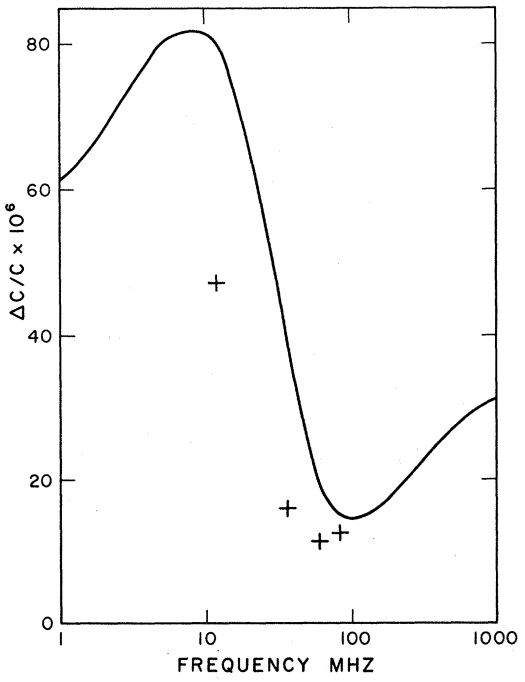

FIG. 1. Frequency dependence of the velocity of sound in liquid He^4 at 0.35°K. The solid line is the theoretical result and the experimental points are the measurements of Abraham *et al.*

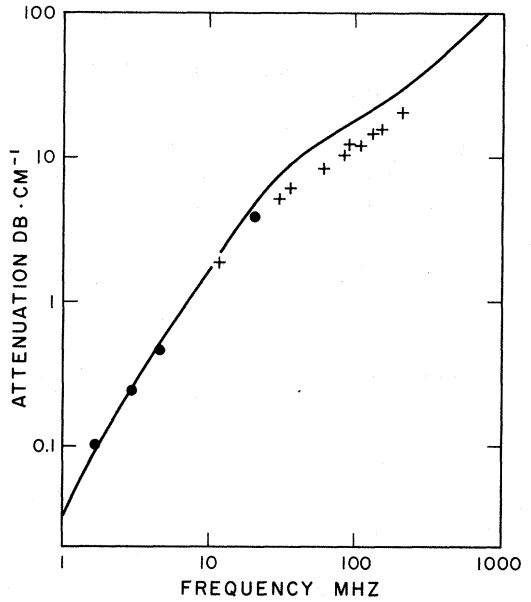

FIG. 2. Frequency dependence of the ultrasonic attenuation in liquid He^4 at 0.35°K. The solid line is the theoretical result. The experimental points at 1.69, 3.0, 4.65, and 20.7 MHz are measurements of Waters, Watmough, and Wilks. The remaining points are measurements by Abraham *et al.*

(Fig. 1). Previous theories with $\gamma > 0$ predict an increase with frequency. (b) The attenuation in the same temperature range depends upon frequency in a complicated way (Fig. 2).

In Fig. 1 we compare theory (solid line) and experiment for the velocity of sound at 0.35°K. The theory correctly predicts the decrease of velocity with frequency in the frequency range which has been investigated experimentally so far. At 36, 60, and 84 MHz the agreement between theory and experiment is very good. At 12 MHz the experimental value is 40% less than theory, but this is probably not serious considering the uncertainties in the theory. In Fig. 2 we present a similar comparison for the attenuation. Here the agreement is extremely good over the entire frequency range for which results are available. Note that the uncertainty in $\partial\epsilon/\partial\rho$ mentioned above is in the right direction to improve the agreement between theory and experiment.

Some predictions of our calculation which could be tested experimentally are the following: (1) For very low frequencies the velocity of sound should increase with frequency. At 0.35°K this should be observed in the frequency range 1 to 10 MHz. (2) In the high-frequency region (above 100 MHz at 0.35°K) the velocity should also increase with frequency. (3) Above 200 MHz the attenuation at 0.35°K should have a linear frequency dependence.

We have also carried out calculations at 0.25 and 0.4°K for a few frequencies. The agreement with experiment is comparable to that obtained at 0.35°K. It is found that features such as the maximum in velocity move to higher frequencies as the temperature is increased. We hope to perform similar calculations using more general forms of the dispersion relation[10] in the near future.

Finally, we note that very different results would be obtained if γ were positive. In this case three-phonon processes become unallowed in first order and the form of the collision integral is completely different. Attempts at explaining the attenuation and velocity using γ positive have not been successful.[1] The good agreement between experiment and our theory constitutes strong evidence that γ is negative and therefore that the dispersion is anomalous.

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dation and the Advanced Research Project Agency.

[1]B. M. Abraham, Y. Eckstein, J. B. Ketterson, M. Kuchnir, and J. B. Vignos, Phys. Rev. 181, 347 (1969).

[2]H. J. Maris and W. E. Massey, Phys. Rev. Lett. 25, 220 (1970).

[3]See, for example, I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965).

[4]N. E. Phillips, C. G. Waterfield, and J. K. Hoffer, Phys. Rev. Lett. 25, 1260 (1970).

[5]G. W. Waters, D. J. Watmough, and J. Wilks, Phys. Lett. 26A, 12 (1967).

[6]Y. Disatnik, Phys. Rev. 158, 162 (1967).

[7]I. M. Khalatnikov and D. M. Chernikova, Zh. Eksp. Teor. Fiz. 50, 411 (1966) [Sov. Phys. JETP 23, 274 (1966)].

[8]B. M. Abraham, Y. Eckstein, J. B. Ketterson, M. Kuchnir, and P. R. Roach, Phys. Rev. A 1, 250 (1970), and 2, 550 (1970).

[9]W. M. Whitney and C. E. Chase, Phys. Rev. 158, 200 (1967).

[10]It is possible that higher-order terms in p may give significant contributions to the dispersion relation for phonons thermally excited at 0.35°K. Also it has been proposed [A. Molinari and T. Regge, Phys. Rev. Lett. 25, 1531 (1971)] that for small p the dispersion relation may be of the more general form $\epsilon(p)=c_0p(1-\gamma_1p-\gamma_2p^2+\cdots)$ with γ_1 negative. These questions will presumably be resolved when specific heat measurements at lower temperatures are made.

[11]Phillips, Waterfield, and Hoffer (Ref. 4) analyzed their specific heat data treating both γ and c_0 as adjustable parameters. They obtained a best fit with $\gamma=-4.1\times10^{37}$ g^{-2} cm^2 sec^2 and $c_0=2.397\times10^4$ cm sec^{-1}. This value of c_0 is considerably outside the experimental uncertainty of the independent measurement of c_0 made in Ref. 9. Consequently we adopted the alternative analysis in which only γ was determined by the specific heat results.

Temporal Behavior of Electron Distributions in an Electric Field

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Coefficients are derived for a one-dimensional Fokker-Planck equation describing the evolution of an electron energy distribution. These coefficients include acceleration of electrons between collisions in addition to the collision terms. The coefficient for the average rate of change of the electron energy, $\langle\Delta\epsilon\rangle/\Delta t$, is the same as obtained with the "average-electron" theory; but the expression for the dispersion, $\langle(\Delta\epsilon)^2\rangle/\Delta t$, has not appeared previously.

The computation of electron energy distributions in an electric field has numerous applications, such as the avalanche breakdown of an insulating gas in a waveguide and the breakdown caused by a laser beam focused on an optical crystal.[1] Traditionally this problem has been treated by a Legendre expansion of the distribution function in three-dimensional velocity space.[2-4] However, practical considerations require truncating the expansion after the first terms, and mathematical complications have limited this method to essentially time-independent problems. The objective of this work is a derivation of the coefficients of a time-dependent Fokker-Planck operator, differing from previous Fokker-Planck operators by including the acceleration of electrons between collisions.

Let $f(\epsilon,t)d\epsilon$ be the expected number of electrons with energies between ϵ and $\epsilon+d\epsilon$. For such a one-dimensional function, it is shown[5] that

$$\frac{\partial f}{\partial t}=-\frac{\partial}{\partial\epsilon}\left(\frac{\langle\Delta\epsilon\rangle}{\Delta t}f\right)+\frac{1}{2}\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\langle(\Delta\epsilon)^2\rangle}{\Delta t}f\right), \tag{1}$$

where

$$\langle(\Delta\epsilon)^n\rangle=\int P(\epsilon|\Delta\epsilon,\Delta t)(\Delta\epsilon)^n d(\Delta\epsilon), \tag{2}$$

with $P(\epsilon|\Delta\epsilon,\Delta t)d(\Delta\epsilon)$ equal to the probability that the electron energy will change from ϵ to the range $d(\Delta\epsilon)$ about $\epsilon+\Delta\epsilon$ during the time Δt.

The basic assumptions made in deriving Eq. (1) are that a time increment Δt can be chosen which is long compared to the time between collisions but short compared to the time in which a significant change occurs in the electron energy, and also that $\langle(\Delta\epsilon)^n\rangle$ is proportional to Δt for $n=1$ or 2, but is proportional to Δt^2 or higher orders for n greater than 2.

Our derivation of the Fokker-Planck coefficients starts with the definition of the probability $P_n(t_1,\cdots,t_n)dt_1\cdots dt_n$ that the electron will have exactly n collisions between t and $t+\Delta t$, and that the first will occur between t_1 and t_1+dt_1, the second between t_2 and t_2+dt_2, etc. If ν is the effective collision frequency, then $P_n(t_1,\cdots,t_n)=\nu^n e^{-\nu\Delta t}$, where the t_j are ordered times.