Spectrum and Anomalous Resistivity for the Saturated Parametric Instability*

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A wave spectrum peaked in angle and broadened in wave number is found from a nonlinear saturation theory for the decay-type parametric instability in the case of nearly equal electron and ion temperatures. The dominant saturation mechanism is nonlinear damping of Langmuir waves by induced scattering from ions. The nonlinear resistivity for transverse waves, including the pump, is obtained from the related mechanism of conversion of transverse into longitudinal waves due to interaction with ions.

The linear theory of parametric instabilities is by now well understood.[1-6] The outstanding theoretical problem is to understand how the level of excited wave fluctuations is saturated by nonlinear wave-wave and wave-particle interactions. A nonlinear saturation theory of the instability in the case $T_e = T_i$ was recently developed by the present authors.[7] Similar independent study has been carried out by Valeo, Perkins, and Oberman.[8] In this Letter, more detailed results of our theory are presented, and the nonlinear conductivity in the presence of the instability is calculated.

As described in earlier papers,[1-6] the role of the exciting field or pump, $E_0 \sin\omega_0 t$, is to couple the Langmuir waves and ion acoustic waves to produce new normal modes, some of which may become unstable. In this communication, we consider only the saturation of the decay[2] branch of the parametric instability, since it occurs over a range of wave numbers disjoint from that of the oscillating, two-stream instability.[5] The new modes in the decay instability consist of components near the Langmuir frequency $\omega_L(k)$ and the acoustic frequency $\omega_a(k)$, each with equal damping rates, and denoted, respectively, by subscripts 1 and 2. There is one set of marginally stable modes (ω_1, ω_2) which become unstable for E_0 above a threshold E_c and another set of modes $(\tilde{\omega}_1, \tilde{\omega}_2)$ which are *more* heavily damped than in the case $E_0 = 0$. The instability threshold for the ω_1, ω_2 modes is[3]

$$E_c^2 = 4\pi n\Theta \times 16 \frac{\gamma_L(k)}{\omega_L(k)} \frac{\gamma_a(k)}{\omega_a(k)} [f(k)\mu^2]^{-1}, \quad (1)$$

where Θ is the temperature in energy units, n is the electron density, $\gamma_L(k)$ and $\gamma_a(k)$ are the damping rates of Langmuir waves and acoustic waves, and $\mu = \hat{k}\cdot\hat{E}_0$. Here $f(k)$ is a resonance function which measures the difficulty of exciting wave numbers k for which the frequency mismatch $\Delta\omega(k) = \omega_0 - \omega_L(k) - \omega_a(k)$ is nonvanishing:

$$f(k) = [1 + (\Delta\omega/2\gamma_a)^2(\omega_a + \delta)^2/\omega_a\delta]^{-1}, \quad (2)$$

where $\delta = \omega_0 - \omega_L(k) = \frac{3}{2}(k_c^2 - k^2)\omega_p/k_D^2$, and k_c is defined by $\delta = 0$. This form of $f(k)$ is valid for $0 < k < k_c$ and follows directly from Eq. (51) of Ref. 5. In Ref. 7 we used a form of $f(k)$ which is only accurate for $\Delta\omega \ll \gamma_a$, i.e., for k near k_m, the perfectly matched wave number defined by $\Delta\omega(k_m) = 0$. To obtain (2), a four-mode coupling scheme[4,5] must be used. The decay instability is restricted to $k < k_c$, and the "oscillating two-stream" instability to $k > k_c$. For equal electron and ion temperatures, the ratio γ_a/ω_a in (1) can be taken to be unity. In what follows, nonlinear effects in γ_a will be ignored because the ion waves are already strongly damped.[9]

Near the instability threshold, the spectral intensity of longitudinal electric field fluctuations, $I(\vec{k}, \omega)$, will be dominated by resonances near the marginally stable mode frequencies, so we can write, to a good approximation,

$$(1/4\pi)I(\vec{k}, \omega) = I_1(\vec{k})\delta(\omega^2 - \omega_1^2)\omega_1 + I_2(\vec{k})\delta(\omega^2 - \omega_2^2)\omega_2.$$

It is shown in the approach of Ref. 7, and can be shown on the basis of a rigorous kinetic theory approach to be presented elsewhere, that the intensities in the high- and low-frequency components of the marginally damped mode for sufficiently slow time variations obey kinetic equations of the form

$$\partial I_{1,2}(\vec{k})/\partial t = -2\gamma_P, \tag{3}$$

where the nonlinear parametric growth rate $-\gamma_P^{NL}$ is given by

$$\gamma_P^{NL} = \gamma_L(k) + \gamma^{NLW}(k) - P\gamma_L(k)f(k)\mu^2, \tag{4}$$

with $P \equiv (E_0^2/E_c^2)_{f=\mu=1}$. The spontaneous term S_1^{NL} has the form

$$S_1^{NL} = 4\pi\Theta\gamma_L(k)\left[1+\frac{\mu^2 Pf(k)}{\omega_0-\omega_1(k)}\omega_p\right]+S_1^{MC}, \tag{5}$$

and a similar result for S_2^{NL} can be given.[3] The term proportional to P is the contribution of spontaneous Cherenkov emission, at the acoustic frequency, mixing with the pump to act as a source for Langmuir waves.[3] It dominates the usual linear emission for $\mu \cong 1$ and $k \cong k_m$ and clearly cannot be ignored. The nonlinear effects are included in $\gamma^{NLW}(\vec{k})$ and $S_1^{MC}(\vec{k})$. We derived expressions for these quantities in Ref. 7. General formulas were also given previously by one of the authors[6]:

$$\gamma_{(k)}^{NLW} = \frac{-e^2}{2m^2}\frac{k_D^4}{\omega_1^3}\int\frac{d^3\bar{k}}{(2\pi)^3}\frac{d\bar{\omega}}{2\pi}(\hat{k}\cdot\hat{\bar{k}})^2\left[\frac{1}{(\vec{k}-\vec{\bar{k}})^2}\,\mathrm{Im}\,\frac{1}{\epsilon^{NL}(\vec{k}-\vec{\bar{k}},\omega_1(k)-\omega)}\right]I(\vec{\bar{k}},\bar{\omega}). \tag{6}$$

The expression for S_1^{MC} is of the same form with the factor in square brackets replaced by $I(\vec{k}-\vec{\bar{k}}, \omega_1(k)-\bar{\omega})(\vec{k}-\vec{\bar{k}})^2$.

Here, $\epsilon^{NL}(\vec{k},\omega)$, defined in Ref. 3, is proportional to the determinant of the mode-coupling matrix. In the derivation of Eq. (3), only the dominant nonlinear corrections to the *diagonal* terms in the mode-coupling matrix have been retained. These diagonal corrections, corresponding to terms of the form of Eq. (6), as well as frequency shifts proportional to $\mathrm{Re}[(\epsilon^{NL})^{-1}]$ should be included in the (implicit) definition of ϵ^{NL}. In the present work, nonlinear frequency shifts *and* the linear shifts which arise when $\Delta\omega \neq 0$ are ignored.[3,6] The form of γ^{NLW} derived in Ref. 7 can be obtained from Eq. (6) by approximating $\mathrm{Im}[(\epsilon^{NL})^{-1}]$ by suitably normalized δ functions for each root ω_S of the dispersion relation.[3,7]

We study only the steady-state solution of (3)–(6). In this case $\partial I_{1,2}/\partial t = 0$ and, provided $\gamma_P^{NL} \ll \gamma_a$, we have

$$I_2/I_1 = (\omega_a/\gamma_a)^2(E_0^2/64\pi n\Theta)(k^2/k_D^2)f(k)\mu^2.$$

Even in the optimum case $\mu^2 f = 1$, this ratio is $<10^{-3}$ in most cases of interest (for $T_e \approx T_i$), so the contribution from I_2 in (6) can be neglected. The remaining contribution to γ^{NLW} represents the damping of Langmuir waves due to induced scattering by ion density fluctuations.[7,10] On carrying out the integrations,[7]

$$2\gamma^{NLW} \cong 2\gamma^{SP} = (\alpha/384\pi^3)(k_D^3/n)(k^2/k_D^2)\omega_p\Theta^{-1},$$
$$\int_0^{2\pi}d\bar{\varphi}\int_{-1}^{1}d\bar{\mu}\,\bar{\psi}(1-\tfrac{1}{2}\bar{\psi}^2)[I_1(k-\tfrac{1}{3}\alpha k_D\bar{\psi},\bar{\mu})-I_1(k+\tfrac{1}{3}\alpha k_D\bar{\psi},\bar{\mu})], \tag{7}$$

where $\bar{\psi} = \sqrt{2}\{1-\mu\bar{\mu}-[(1-\mu^2)(1-\mu^{-2})]^{1/2}\cos\bar{\varphi}\}^{1/2}$. If $I_1(k,\mu)$ is a sufficiently slowly varying function of k, we can replace[7] the difference of I's by $\frac{2}{3}\alpha k_D\bar{\psi}\,\partial I_1/\partial k$. It follows from (3), (4), (5), and (7), with the use of the derivative approximation, that the angular dependence of $I_1(k,\mu)$ must be of the form

$$I_1(k,\mu) = 4\pi\Theta[1+Pf(K)\mu^2/\alpha K][a(K)-b(K)\mu^2]^{-1}, \tag{8}$$

where $a(K)$ and $b(K)$ satisfy

$$a(K) = 1-\tfrac{3}{2}d(\partial/\partial K)\int_{-1}^{1}d\bar{\mu}\,K^2 I_1(K,\bar{\mu})(1-\bar{\mu}^2)\Theta^{-1},$$
$$b(K) = Pf+\tfrac{3}{2}d(\partial/\partial K)\int_{-1}^{1}d\bar{\mu}\,K^2 I_1(K,\bar{\mu})(3\bar{\mu}^3-1)\Theta^{-1}, \tag{9}$$

with $d = (\alpha^2/432\pi^2)(k_D^3/n)\omega_p/\gamma_L \cong 2\times10^{-3}\alpha^2$, and $K = k/k_D$. These equations have been solved numerically for various values of P with K in the interval $0 \leq K < K_c$ with the boundary conditions $a(K_c) = 1$, $b(K_c) = 0$. In the neighborhood of K_m for $P \gtrsim 1$, b approaches 1 with a infinitesimally greater than b, giving rise to a spectrum very sharply peaked parallel and antiparallel to $\vec{E}_0$. In this region, with $a \cong 1$, analytic formulas have been obtained for b and the angle-averaged spectrum $\bar{I}_1(k)$:

$$b = 1 - 4\exp\left[-\frac{\bar{I}_1(k)/\Theta}{2\pi(1+Pf/\alpha K)}\right], \tag{10}$$

$$\bar{I}_1(k) \equiv \tfrac{1}{2}\int_{-1}^{+1} d\mu\, I_1(\mu,k) = (\Theta/6dK^2)\int_K^{K_c}[Pf(\bar{K})-1]d\bar{K}. \tag{11}$$

These formulas are valid over a range of wave numbers such that b is close to 1. In Fig. 1, $\bar{I}_1(k)/\Theta$ is plotted for $P=5$, 10, and 20, with $\alpha = 6\times10^{-3}$ and $K_m = 0.2$. The $\bar{I}_1(k)$ spectra have about one half the peak value and are about three times as broad as the results of Ref. 7, where the assumption of an isotropic spectrum in μ was made. The width w_I of $\bar{I}_1(k)$ and the total energy $\bar{E}$ are given by the approximate analytic formulas ($P>1$)

$$w_I \approx P\int_{K_m}^{K_c} f(K) \approx P\tfrac{2}{3}\alpha,$$
$$\bar{E} \equiv \int_0^{k_c} \frac{dk\, k^2}{8\pi^3}\,\bar{I}_1(k) \approx \frac{2}{\pi}\, n\Theta \frac{\gamma_L}{\omega_p}(P^2 - P). \tag{12}$$

The spectrum spreads towards smaller K out of the active region around K_m, as predicted by the global arguments of Ref. 7. $\bar{E}$ is of the same order as found there. Valeo, Oberman, and Perkins, using a different formal approach, have independently obtained similar results.[8] Because of the very rapid rise of $b(K)$ and, therefore, $I(k,\mu)$ (for $\mu\approx1$) in the region $K_m<K<K_c$, the derivative approximation to (9), upon which the above numerical results are based, is not valid in this region. The complete form of (9) should be used here. The resulting integral-difference equation has not yet been solved.

The damping of a *transverse* wave of frequency $\omega_T \neq \omega_0$, in the presence of the enhanced fluctuations parametrically excited by the pump, is calculated in complete analogy to γ^{NLW}. The analysis of this expression to dominant terms follows as before, however, with the long wavelength and polarization appropriate to a transverse wave. We obtain the expression analogous to (7):

$$\frac{2\gamma_T^{NL}}{\omega_p} = 8\pi\frac{\sigma_T^{NL}(\omega_T)}{\omega_P} = \frac{\alpha}{384\pi^2}\frac{k_D^2}{n}\frac{k_m^2}{k_D^2}\Theta^{-1}\int_{-1}^{1} d\bar{\mu}\,\bar{\mu}^2[I_1(\tilde{k}_m,\bar{\mu}) - I_1(\tilde{k}_m + \tfrac{1}{3}k_D\alpha, \bar{\mu})], \tag{13}$$

where $\tilde{k}_m/k_D = [(\omega_T^2-\omega_P^2)/3\omega_P^2]^{1/2} - \alpha/3$. This reduces to the k_m defined following (2) if $\omega_T = \omega_0$. Here, we have also defined an effective nonlinear conductivity $\sigma_T^{NL}(\omega_T)$. This formula applies to a weak transverse wave in the presence of the enhanced fluctuations, such as the diagnostic wave in the radar experiments of Cohen and Whitehead.[11,12]

The nonlinear conductivity of the pump wave itself can be written in the form (for $P\gg1$)

$$\pi\sigma_T^{NL}(\omega_0)|E_0|^2 = \int d^3k\,(2\pi)^{-3}Pf(k)\mu^2\gamma_L(k)I_1(k). \tag{14}$$

The power lost by the pump due to the parametric coupling therefore goes completely into the power gain of plasma waves arising from the negative damping contribution of γ_P^{NL} [Eq. (4)]. Since the nonlinear interactions conserve Langmuir plasmons[7] they therefore nearly conserve Langmuir wave energy if $k\ll k_D$. It then follows that the total power balance in a volume element of the system can be written

$$\nabla\cdot\vec{S} = \tfrac{1}{2}\sigma_l(\omega_0)|E_0|^2 + 2\int\frac{d^3k}{(2\pi)^3}\gamma_L(k)\frac{I_1(k)}{4\pi}, \tag{15}$$

where $\sigma_l(\omega_0)$ is the *linear* conductivity of the pump. Thus the net (Poynting) power flux $\nabla\cdot\vec{S}$ flowing into the volume element is dissipated by the *linear* losses of the pump and the Langmuir waves.[3,6] If $k_m\ll k_D$, the wave dissipation is entirely collisional.

These formulas for longitudinal and transverse nonlinear dampings or conductivities are proportional to an integral over I_1, the Langmuir frequency part of the wave fluctuation spectrum. This is in distinction to the well-known Dawson-Oberman[13] (DO) conductivity formula which is proportional to the low-frequency ion density fluctuation spectrum. The terms which reduce to the DO formula are those terms proportional to I_2, which were neglected because of the small ratio I_2/I_1. Our σ_T^{NL} expression is proportional to $\alpha = (m_e/m_i)^{1/2}$ and is usually neglected in studying low-frequency instabilities. The terms proportional to α and the high-frequency spectral intensity were studied in the linear case by Nishikawa and Ichikawa.[13] For $T_e\approx T_i$ the interpretation of Kaw and Dawson[15] of the anomalous conductivity on the basis of the DO formula must be revised. However, for $T_e\gg T_i$ the ratio I_2/I_1 can

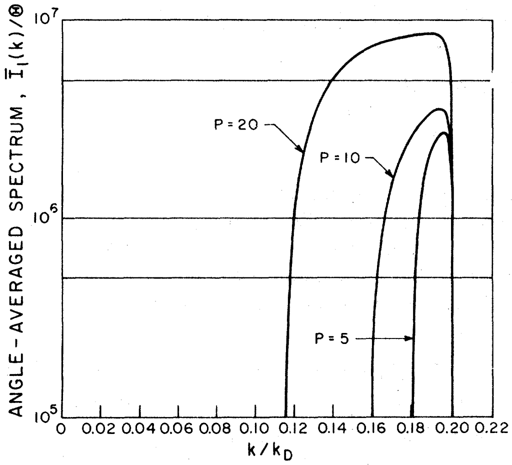

FIG. 1. Angle-averaged spectrum of Langmuir frequency electrostatic field fluctuations, $\bar{I}_1(k)$, for $\alpha = (m_e/m_i)^{1/2} = 6\times 10^{-3}$, with $P \equiv (E_0^2/E_0^2)_{f=\mu=1} = 5$, 10, and 20 assuming a frequency-matched wave number $k_m/k_D = 0.2$.

be of order unity or greater because the factor $(\omega_a/\gamma_a)^2$ in (8) becomes large. Both low- and high-frequency wave contributions may then have to be retained.

From (15) we have estimated $4\pi\sigma_T^{NL}(\omega_0) \simeq (P/6)\gamma_L$ when $P \gg 1$. In typical laser-plasma experiments the value of P can be 60 or larger, leading to an enhanced differential absorption coefficient as much as 10 or more times the linear value. This occurs over a range of distances into the density profile such that the excited plasma waves are collisionally damped. The requirement is that k_m be <0.2, which corresponds to roughly 10% of the density scale length. In such cases the total absorption and the propagation of the pump wave must be determined self-consistently by taking into account the depletion of the pump resulting from the large value of $\sigma_T^{NL}(\omega_0)$. This is properly done by solving a nonlinear Maxwell equation for E_0 in a plane-layered plasma.

The spectra observed[12] by incoherent scattering in the ionosphere cannot be as sharply peaked in the pump direction as in the present theory because sizable enhancements at 38° to this direction are found. It is likely that geomagnetic field and wave convective effects play an important role there. These matters will be discussed in a future publication.

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