Total Binding Energies of Nuclei, and Particle-Removal Experiments*

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Under the assumptions of the impulse approximation, there is a relation between the cross section for removing a particle from a system and the total binding energy of the system, if it is bound by two-body forces. This relation is established and applied to the $(p,2p)$ reaction on nuclear targets. Data from recent experiments are used to obtain values for the total binding energy of the protons in various nuclei, which are then compared with the measured values.

This paper is concerned with the hole energies in nuclei, measured in a direct reaction, such as $(p,2p)$, in which one nucleon is removed from a nuclear target, and with the total energy of the target. For the reaction $A(p,2p)B^*$, one measures the differential cross section as a function of the momenta $\vec{k}_1$ and $\vec{k}_2$ of the emerging protons. The state of the final nucleus B^* is specified by two quantities:

$$\vec{k}=\vec{k}_1+\vec{k}_2-\vec{k}_0,\qquad E=E_0-E_1-E_2-E_R, \tag{1}$$

where $\vec{k}_0$ and E_0 are the momentum and energy of the initial proton, and E_1, E_2, and E_R are the kinetic energies of the two emerging protons, and of the recoiling nucleus B^*, respectively.

In the usual plane-wave impulse approximation,[1] the cross section for $(p,2p)$ takes the form

$$d^6\sigma/d^3k_1d^3k_2=g\sigma_{pp}(\vec{k}_0,\vec{k}_1,\vec{k}_2)P(\vec{k},E)\times\delta(E-E_0+E_1+E_2+E_R), \tag{2}$$

where σ_{pp} is the p-p differential cross section and g is a kinematical coefficient. The function $P(\vec{k},E)$ gives the probability that if a proton of momentum $\vec{k}$ is removed from the target A, the final target B is left with excitation energy E^* relative to its ground state, or $E=E^*-Q_0$ relative to the ground state of A, where $Q_0=E_A-E_B$ (ground-state energies).

Now there is an interesting relation, in the form of a sum rule, between $P(\vec{k},E)$ and the total binding energy of the protons in the target, which I shall discuss and apply to some recent $(p,2p)$ experiments.[2] For what follows, it is necessary to assume that Eq. (2) is correct. It is expected that there are corrections from distortions of the proton waves, but calculations have shown[3] that the largest effect is a reduction of the overall magnitude. This effect appears to be slowly varying in $\vec{k}$ and E, and we shall simply absorb it into the constant g. In principle, if one could calculate the distortion effects properly, one could extract the undistorted $P(\vec{k},E)$.

We can define $P(\vec{k},E)$ as the expectation value in the target ground state (labeled A) of an operator composed of $a(\vec{k})$ and $a^\dagger(\vec{k})$ which remove or add a proton with momentum $\vec{k}$:

$$P(\vec{k},E)\equiv\langle A|a^\dagger(\vec{k})\delta(E-H)a(\vec{k})|A\rangle=\sum_f|\langle f|a(\vec{k})|A\rangle|^2\delta(E-E_f), \tag{3}$$

where H is the nuclear Hamiltonian, and f labels the final state of the nucleus B, with $E=E_f$. For an unpolarized (or spin-0) target, P depends only on $|\vec{k}|$, so we write $P(k,E)$.

More generally one can define $P(\lambda,E)$ for any complete set of single-particle states (for example, a set of shell-model orbitals), labeled by λ, by replacing $a(\vec{k})$ by $a(\lambda)$ in (3), and so on. Then the occupation of an orbit is given by

$$n(\lambda)\equiv\int_{-\infty}^{\infty}P(\lambda,E)\,dE=\langle A|a^\dagger(\lambda)a(\lambda)|A\rangle, \tag{4}$$

and the removal energy for λ by

$$\epsilon(\lambda)=-\int_{-\infty}^{\infty}P(\lambda,E)E\,dE=\langle A|a^\dagger(\lambda)[a(\lambda),H]|A\rangle. \tag{5}$$

The *mean* removal energy, which is obtained by normalizing (5),

$$E_\lambda\equiv\epsilon(\lambda)/n(\lambda), \tag{5a}$$

is a convenient definition of the hole energy for λ.

There is a sum rule relating $\epsilon(\lambda)$ with the total binding of the target, E_A, and with the kinetic energy of the target. The relation is not new, and is implied in several works on Green's functions.[4,5] Since an explicit formulation and simple derivation do not seem to be available in the literature, they shall be given here.

It is assumed that the Hamiltonian involves interactions involving two, but not more, particles.

We write $H = H_1 + H_2$ with one- and two-body terms,

$$H_1 = \sum_{\alpha,\beta} \langle \alpha | H_1 | \beta \rangle a^\dagger(\alpha) a(\beta),$$
$$H_2 = \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | H_2 | \gamma\delta \rangle a^\dagger(\alpha) a^\dagger(\beta) a(\delta) a(\gamma). \tag{6}$$

Then, from (5), we obtain

$$\epsilon(\lambda) = \sum_\beta \langle \lambda | H_1 | \beta \rangle \langle A | a^\dagger(\lambda) a(\beta) | A \rangle + \sum_{\beta\gamma\delta} \langle \lambda\beta | H_2 | \gamma\delta \rangle \langle A | a^\dagger(\lambda) a^\dagger(\beta) a(\delta) a(\gamma) | A \rangle. \tag{7}$$

Summing over all states λ,

$$\sum_\lambda \epsilon(\lambda) = \langle A | H_1 | A \rangle + 2 \langle A | H_2 | A \rangle. \tag{8}$$

Since the total energy of the target A is $E_A = \langle A | H_1 + H_2 | A \rangle$, we obtain the desired sum rule,

$$E_A = \tfrac{1}{2} \langle A | H_1 | A \rangle + \tfrac{1}{2} \sum_\lambda \epsilon(\lambda). \tag{9}$$

Note that (9) is independent of the choice of the set of states $\{\lambda\}$. In the absence of external interactions, H_1 is the kinetic energy. Clearly, for a many-body Hamiltonian of the form $H = \sum_n H_n$, $\sum_\lambda \epsilon(\lambda) = \sum_n n \langle A | H_n | A \rangle$, and (9) does not follow.

For the case of the $(p, 2p)$ experiments, we take $\{\lambda\}$ to be the momentum states $\{\vec{k}\}$. Since only protons are removed, (9) must be modified as follows. We write the Hamiltonian in a proton-neutron (p, n) notation:

$$H = H_1(p) + H_1(n) + H_2(pp) + H_2(nn) + H_2(pn).$$

Then (9) takes the form

$$E_Z = \tfrac{1}{2} \int_{-\infty}^{\infty} dE \int d^3k \, (k^2/2m - E) P(k, E), \tag{10}$$

where E_Z is the total energy of the protons in A, is given by

$$E_Z = \langle A | H_1(p) + H_2(pp) + \tfrac{1}{2} H_2(pn) | A \rangle. \tag{11}$$

There is a similar sum rule relating the neutron removal energy to the total neutron energy E_N.

The total energy of the target $E_A = E_N$ is a directly measured quantity. To obtain E_Z we must also have the total Coulomb energy and the symmetry energy, which are also measured but less directly, by using the energies of ground or analog states of neighboring nuclei. If we assume a "mass formula" for the energy,

$$E_A = \alpha(A) + \beta(A)(N - Z)^2 + \gamma(A, Z), \tag{12a}$$

where β gives the symmetry energy and γ the Coulomb energy, then we find

$$E_Z = (Z/A)[E_A + 2\beta N(Z - N) + (N/Z)\gamma]. \tag{12b}$$

The only restriction in (12a) is that $N - Z$ enters as $(N - Z)^2$.

The recent $(p, 2p)$ experiments of James *et al.*,[2] performed with 385-MeV protons, provide a set of measurements of $P(k, E)$ for a variety of nuclei, to which we can apply the sum rule (10). We calculate the following quantities directly from the experimental $P(k, E)$:

$$T_m = n^{-1} \int dE \int d^3k \, (k^2/2m) P(k, E),$$
$$E_m = n^{-1} \int dE \int d^3k \, (-E) P(k, E),$$
$$n = \int dE \int d^3k \, P(k, E),$$
$$(E_Z/Z)_m = \tfrac{1}{2} T_m + \tfrac{1}{2} E_m. \tag{13}$$

The limits of integration are the limits of the reported data: $0 \le k \le 1.2\ \mathrm{fm}^{-1}$; and E runs from $-Q_0$ (5–10 MeV) to 65–70 MeV. The quantities T_m and E_m give, respectively, the mean kinetic energy and removal energy, per proton, and are listed along with $(E_Z/Z)_m$ in Table I for the targets ^{12}C, ^{40}Ca, ^{120}Sn, and ^{208}Pb. The normalization n, which would equal the number of target protons (Z) if there were no absorption of the scattering protons, is in fact reduced by a factor (see Table I of Ref. 2) which runs from ~0.2 for ^{12}C to ~0.02 for ^{208}Pb. Again, we have treated the absorption as if it were independent of kinematics, which may be a reasonable approximation[3] at 385 MeV.

The total energy of the protons is given per

TABLE I. Comparison of the total energy (E_Z/Z) of the protons per proton, calculated in two ways from the $(p, 2p)$ experiments of Ref. 2, with the measured values for several target nuclei. The momentum sum rule uses (13), the orbital analysis is from Ref. 2, and the measured value uses (12), all in MeV.

	Momentum sum rule			Orbital sum rule			Experiment
Target	T_m	E_m	$(E_Z/Z)_m$	T_{orb}	E_{orb}	$(E_Z/Z)_{\mathrm{orb}}$	$(E_Z/Z)_{\mathrm{expt}}$
^{12}C	14.1	−28.2	−7.05	11	−22	−5.5	−6.93
^{40}Ca	13.8	−28.0	−7.1	12.3	−24.5	−6.1	−6.73
^{120}Sn	14.4	−30.2	−7.9	16.8	−25.4	−4.3	−9.32
^{208}Pb	14.3	−27.7	−6.7	···	···	···	−8.22

proton as $(E_Z/Z)_{\rm expt}$ in Table I. The values are obtained from nuclear binding energies[6] and Coulomb energies,[7] by using (12b), with the symmetry energy coefficient $\beta(A)=23.7$ MeV/A.

The sum-rule values $(E_Z/Z)_m$ are in reasonable agreement with the binding energies $(E_Z/Z)_{\rm expt}$. We have not assigned an uncertainty in the sum rule due to experimental uncertainty, but it could easily be ± 2 MeV. In addition, the data strongly suggest that the kinematic limits of $k<1.2$ fm^{-1}, $E<80$ MeV, may not cover all contributions to $P(k,E)$. This limitation is quite likely more serious for the heavier nuclei, which may explain why the agreement is closer for C and Ca than for Sn and Pb.

There is a new set of $(p,2p)$ experiments[8] that were performed at 600 MeV on a variety of targets, which would also be interesting to analyze in the same way. Unfortunately, these measurements concentrated on smaller values of k only, so that a direct application of the sum rule (10) is not possible with the published data.

James *et al.* have treated their own data in a completely different way; they assume that every proton can be associated with a shell-model orbit, which would be normally filled in a Hartree-Fock picture. They decompose $P(k,E)$ into contributions from each filled orbit, using harmonic-oscillator momentum distributions, as described in their papers.[2] For high E, $P(k,E)$ is not well represented by oscillators and is neglected as "background."

One may calculate from this orbital analysis values of the mean kinetic energy $T_{\rm orb}$ and removal energy $E_{\rm orb}$ analogous to (13), as well as $(E_Z/Z)_{\rm orb}=\frac{1}{2}(T_{\rm orb}+E_{\rm orb})$. We include these quantities in Table I for the three lighter nuclei for which the orbital separation can be performed. This approach gives systematically less binding per proton than does the momentum sum rule (10), although based on the same data. This is not only because of the neglect of the high-E part of $P(k,E)$, which is not fitted by normally occupied orbitals. The orbital analysis also fixes the number of protons in each orbit, while $P(k,E)$ shows *apparently* more protons in high-E orbits than there would be in the shell model, again increasing $|E_m|$ over $|E_{\rm orb}|$.

For the orbital analysis, one should really use the sum rule in the form (9) (but for protons), and the result should be the same as for (10) *if one has included all orbitals.* Presumably the target is not a Hartree-Fock nucleus, and many higher orbits are fractionally occupied. The "background" term in the James analysis may include these orbits, and should therefore not be neglected. Similarly, the states of high E (~ 80 MeV) seen in $(e,e'p)$ reactions by Amaldi *et al.*[9] may also represent protons in high orbits.

On the other hand, these comparisons only make sense to the extent that one can trust the extraction of $P(k,E)$ from the cross sections. The corrections to the assumed form (2) could turn out to be strongly dependent on k or E. For this one must know more about the theory of the scattering process, including corrections not included in the optical distortion of the protons. It would be useful to have experiments at other energies[8] and with other projectiles, such as $(e,e'p)$[9] and $(\pi,\pi'p)$, but over the entire kinematic range of k and E.

Finally, we note that the orbital removal energy $\epsilon(\lambda)$ defined in (5) has also recently been discussed by Shakin and Da Providência[10] in connection with the Brueckner theory of nuclei. Baranger[11] has discussed a similar orbital energy, which differs from $\epsilon(\lambda)$ by the inclusion of contributions from the *addition* of a particle to orbit λ. In a separate paper we shall further develop the theory of the removal energy in the context of linked-cluster perturbation theory, to make further contact with the theory of the nuclear ground state.

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Differential Cross Sections for Small-Angle Neutron-Proton and Neutron-Nucleus Elastic Scattering at 4.8 GeV/c[*]

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Measurements of the small-angle elastic scattering of 4.8-GeV/c neutrons from hydrogen and heavier nuclei are reported over the range of four-momentum transfer between 0.002 and 0.05 (GeV/c)^2. The data were used to determine the slope of the small-$|t|$ diffraction peak. The incident neutron beam covered a broad energy spectrum, and the detector included an ionization calorimeter to provide a measurement of the neutron energy.

Elastic pp scattering at small momentum transfers includes contributions from both Coulomb scattering and strong interactions. At incident momenta of a few GeV/c, Coulomb effects are important for $|t| \lesssim 0.01$ (GeV/c)^2, and are relatively more important for proton scattering on heavier nuclei. In view of the narrow peak in np scattering at 180°, it is of interest to explore the elastic scattering at very small $|t|$ in the np system where Coulomb effects may be neglected.

This experiment, at the Bevatron of the Lawrence Berkeley Laboratory, utilized a neutron beam produced at 0° by protons on an internal beryllium target. The beam was cleared of γ rays by 5 radiation lengths of lead, and was defined by a collimator of 1.6 cm diameter and 2.6 m length. Charged particles were removed by the Bevatron field and two sweeping magnets. The cryogenic liquid-hydrogen target was 122 cm in length, and was surrounded by anticoincidence counters to veto inelastic interactions which produced fast charged particles and γ rays. Solid targets of carbon and metals were typically about 0.3 interaction-mean-free-paths thick, and made use of the same anticoincidence counters.

The neutron detector consisted of an iron converter plate followed by an X-Y coordinate counter system and an ionization calorimeter. The ionization calorimeter, consisting of fourteen 0.6-cm-thick scintillators interspersed with thirteen 3.8-cm aluminum plates, had a sensitive area of 30×56 cm^2. The 2.5-cm Fe plate converted about 20% of the incident neutrons; the resulting charged conversion products were detected in horizontal and vertical scintillation counters each viewed from both ends by photomultiplier tubes. By measuring the time-of-flight difference of scintillation light to these tubes, horizontal and vertical (X-Y) coordinates of the neutron conversion were determined. The effective uncertainty in the position of a neutron interaction (including uncertainty due to beam size and the spreading inherent in the conversion process) was ± 2.7 cm.

For each detected event, the pulse height from the summed signals of the fourteen calorimeter counters was digitized and recorded along with the digitized X-Y conversion coordinates. Data were taken with the targets in and out of the beam. A hole 5.08 cm in diameter in the converter iron plate and subsequent aluminum plates reduced the conversion probability for beam neutrons. These beam neutrons produced a background due to interactions in the calorimeter which produced charged particles going backward into the X-Y counters, thus mimicking a scattered neutron. Despite considerable efforts to reduce this effect, it caused a large background which limited the signal-to-background rate to about 1:1 in a typical angular interval for the hydrogen target. Since this background was proportional to the beam flux transmitted through the target,