Collective Modes in Streams of Charged Vortex Rings

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It is shown that a stream of charged vortex rings supports collective modes which are unstable below a critical wave number, thus leading to a charge-density wave structure. The negative effective mass of a gas of vortex rings plays a crucial role in the instability. For a singly charged stream, the situation is analogous to that of gravitational collapse.

Charged particles moving in superfluid helium[1] at sufficiently high velocities create vortex rings and get attached to them with an energy of the order of 20 K. Through experiments with charged vortex rings, Rayfield and Reif[2] verified the energy-velocity relation of vortex rings,

$$v=\frac{\kappa}{4\pi R}\left(\ln\frac{8R}{a}-\frac{1}{4}\right), \quad E=\tfrac{1}{2}\rho_s\kappa^2R\left(\ln\frac{8R}{a}-\frac{7}{4}\right), \quad (1)$$

where κ is the circulation, R the radius, a the core radius, and ρ_s the superfluid density. It has recently been found[3] that a pulse of charged vortex rings achieves a steady-state distribution after traveling some distance indicating the effect of mutual interactions among the vortices. Here we investigate the collective mode of oscillation of beams of charged vortices arising from these interactions.

We take the charged vortex rings to be quasiparticles interacting with each other through the Coulomb[4] and hydrodynamic dipolar forces. If we consider a unidirectional beam of charged vortex rings of a diameter large compared to the size of a ring, the hydrodynamic forces may be neglected.[5] For simplicity, we shall ignore for dynamical purposes the slowly varying logarithmic terms in the energy-velocity relation (1), and make an approximation in the numerical factors so that we have

$$v \cong \beta/E, \quad \beta \simeq \tfrac{1}{2}\rho_s\kappa^3/4\pi. \quad (2)$$

Correspondingly, we have the impulse velocity relation

$$p \simeq \alpha/v^2, \quad \alpha \simeq (\rho_s\kappa^3/8\pi)\ln(8R_0/a), \quad (3)$$

where R_0 is the average radius of the vortices.

First we consider a stream of singly charged vortices. Let $f(x,p,t)$ be the phase-space distribution of the vortices. The one-dimensional Vlasov equation for $f(x,p,t)$ is[6]

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}+qE\frac{\partial f}{\partial p}=0, \quad (4)$$

where

$$\partial E/\partial x=4\pi qn, \quad (5)$$

and q is the charge on the vortex; f relates with n through

$$n(r,t)=n_0\int f(x,p,t)\,dp, \quad (6)$$

where n_0 is the average density. We write

$$f=f^0(p)+f^{(1)}(x,p,t)+\cdots$$

and linearize the Vlasov equation to get

$$\frac{\partial f^{(1)}}{\partial t}+v\frac{\partial f^{(1)}}{\partial x}+qE^{(1)}\frac{\partial f^0}{\partial p}=0, \quad (7)$$

where

$$\partial E^{(1)}/\partial x=4\pi qn^{(1)}, \quad (8)$$

$$n^{(1)}=n_0\int f^{(1)}(x,p,t)\,dp. \quad (9)$$

We assume here for simplicity that the stream is very long in the x direction and the dc electric field may be ignored.[7]

We look for solutions of (7) of the form

$$f^{(1)}(x,p,t)=g(p)e^{i(kx-\omega t)}.$$

We then find from (7) and (8)

$$1-\frac{4\pi q^2n_0}{k^2}\int_{-\infty}^{\infty}\frac{\partial f^{(0)}/\partial p}{v-\omega/k}dp=0, \quad (10)$$

where the integration path is taken below the pole $v=\omega/k$. Equation (10) may be written in terms of the velocity $\Delta v=v-\bar{v}$, where $\bar{v}$ is the mean velocity. We consider that the unperturbed distribution function $f^{(0)}$ is a function of the deviation of the impulse Δp from its average value. A typical form of $f^{(0)}(\Delta p)$ may be given by

$$f^{(0)}(\Delta p)=N\exp[-(\Delta p/p_0)^{1/2}], \quad (11)$$

where N is the normalization factor, and p_0 is the characteristic momentum of the group of vortices. We then treat $\Delta v/\bar{v}$ to be a small quantity and expand (10) in terms of it. We note from (3) that

$$\Delta p=m^*\Delta v, \quad (12)$$

where the equivalent mass m^* (<0) is given by

$$m^* = -2\alpha/\bar{v}^3. \quad (13)$$

If we now write $\omega \equiv \omega_r + i\omega_i$, we obtain the dispersion relation for $\omega \gg \omega_i$,

$$(\omega_r - k\bar{v})^2 \simeq -\omega_0^2 + 3k^2 v_T^2, \quad (14)$$

$$\omega_i \simeq \frac{\pi(\omega_0^2 m^*)^2}{2\sqrt{3}k^3 v_T} \frac{\partial f^{(0)}}{\partial \Delta p}\bigg|_{\Delta p = m^*(\omega_r/k - \bar{v})}, \quad (15)$$

where

$$\omega_0^2 = -4\pi n_0 q^2/m^* > 0, \quad v_T^2 = \langle \Delta v^2 \rangle. \quad (16)$$

We note that to obtain stable waves requires $k > k_c$, where

$$k_c^2 \simeq \omega_0^2/3v_T^2. \quad (17)$$

From (11), we see that $f^{0\prime} < 0$, so that $\omega_i < 0$ indicating Landau damping of the collective mode.

For $k < k_c$, the wave is unstable. We then have that

$$\omega - k\bar{v} = \pm i(\omega_0^2 - 3k^2 v_T^2)^{1/2} \quad (18)$$

or

$$\omega_i \simeq \omega_0.$$

Thus, for $k < k_c$, the waves are exponentially growing in the linear approximation at a rate ω_i^{-1}. We note that this instability is due directly to the negative mass of a gas of vortex rings. For a typical set of experimental parameters we find $\omega_0 \sim 10\ \text{sec}^{-1}$. Thus both the stable and the unstable oscillations should be easily observable experimentally. Note that the dispersion relation obtained in (18) closely resembles that of the gravitational collapse.

Equation (18) indicates that the stream is also convectively unstable in that a spatial amplification is also possible for a fixed value of modulation frequency. This fact can easily be demonstrated by solving Eq. (18) for a complex k for a given real frequency ω to give

$$k = \frac{\omega\bar{v}}{\bar{v}^2 - 3v_T^2}\left\{1 \pm i\left[\frac{\omega_0^2}{\omega^2}\left(1 - \frac{3v_T^2}{\bar{v}^2}\right) - \frac{3v_T^2}{\bar{v}^2}\right]^{1/2}\right\}$$
$$\sim \frac{\omega}{\bar{v}}\left[1 \pm i\left(\frac{\omega_0^2}{\omega^2} - \frac{3v_T^2}{\bar{v}^2}\right)^{1/2}\right]. \quad (19)$$

The spatial amplification is possible for $\omega < \omega_c$, where the critical frequency ω_c is given by

$$\omega_c = \omega_0 \bar{v}/\sqrt{3}\,v_T. \quad (20)$$

The exponential growth of the waves is clearly an artifact of the linearization. For $k = k_c$, we expect a new structure stabilized by the nonlinear terms,[3] which we may identify as a *charge-density wave.*

Next, we examine the case of two counterstreaming oppositely charged vortex-ring beams. By adding the contribution of the additional vortex rings with an average speed $-\bar{v}$ in (10), we obtain the dispersion relation for the real part of the frequency,

$$-1 = \frac{\omega_0^2}{(\omega - k\bar{v})^2 - 3k^2 v_T^2} + \frac{\omega_0^2}{(\omega + k\bar{v})^2 - 3k^2 v_T^2}. \quad (21)$$

For an ordinary plasma an instability can occur under certain conditions due to counterstreaming.[8] Here we may deduce from (21) that the ordinary counterstreaming instability is eliminated. However, the instability due to the negative mass still occurs for $k < k_c$.

If the beam is not unidirectional, the hydrodynamic forces must be considered. In that case besides these longitudinal modes we have investigated, there is the possibility of occurrence of transverse modes.

Experiments by G. Gamota stimulated our interest in this problem. We would like to thank him as well for several discussions.

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[1]For a review see, for example, G. Gamota, J. Phys. (Paris), Colloq. 31, C-3-39 (1970).

[2]G. W. Rayfield and F. Reif, Phys. Rev. 136, A1194 (1964).

[3]G. Gamota, A. Hasegawa, and C. M. Varma, Phys. Rev. Lett. 26, 960 (1971).

[4]This would break down at very short distances.

[5]See footnote 4 of Ref. 3.

[6]We have used the Vlasov equation because our system is very dilute, and we are studying finite-frequency behavior. In the collision-dominated regime the same results can be obtained from the fluid equations with a numerical factor, γ (cf. Ref. 3), multiplying the k^2 term in Eqs. (14) and (18).

[7]Note that there is no dc electric field for a neutral stream of charged particles. We have chosen to present the case of a singly charged beam since it relates to experiment. To obtain the zero-order steady-state distribution, one must now solve a nonlinear problem including the nonuniform dc electric field. The earlier experimental and the theoretical work has shown that a steady-state distribution exists if a pulse of vortices travels for a length large compared to the width. To obtain the collective modes we must consider perturbations over the steady-state distribution. This is a difficult nonlinear problem. We can see, however, that the time to achieve the steady-state distribution is so large under certain circumstances that we may assume

a zero-order spatially uniform distribution. The characteristic time taken to achieve steady state in a system of length L is of the same order of magnitude as the time taken by a given vortex ring to travel the distance L in the presence of the electric field due to the rest. Since $dE = ne^2x\,dx$, using (2) we have that $v(x) = \beta/ne^2x^{-2}$. Thus the time to travel the distance L is $t_{eq}(L) = \frac{1}{3}ne^2\beta L^3$. This is to be compared with the time for the steady flow of vortex rings, $t_{transit}(L) = L/v$. Thus for a long enough system so that $t_{eq}(L) \gg t_{transit}(L)$, the distribution will remain approximately uniform.

[8]For example, T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill, New York, 1962), p. 110.

Growing Collective Modes in Vortex Ring Beams in He II

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Growing collective modes are observed in beams of charged vortex rings in superfluid helium. The growth of the waves is found to be dependent upon the path length of the beam and the modulation frequency, in agreement with theory.

In previous work[1] it has been shown that a pulsed beam of charged quantized vortex rings displays collective behavior and evolves into a steady state. This follows from the counteracting effect of the Coulomb field and negative effective mass of the vortex ring.[2] The mutual interactions that lead to such behavior are conducive to possible excitation of collective modes of oscillation. In the preceding paper[3] Hasegawa and Varma derive the dispersion relation for such waves and predict that the wave amplitude should grow within a certain range of frequencies and be dependent upon the distance travelled. In the present experiment we have observed these collective modes for the first time. To create them we produce a small density perturbation by modulating the energy of the beam and then observe the growth of the density perturbation as a function of frequency and drift space. The results are found to be in agreement with the theoretical predictions.

The experimental cell is immersed in superfluid helium ($T \sim 0.3$ K) and is shown in Fig. 1. Four grids, G_1–G_4, are spaced between a radioactive source and a guarded collector. G_1 is 5.7 mm from the source; G_2 and G_3 are spaced by 0.13 mm and are located between G_1 and G_4, the latter being 1 mm away from the collector C and acting as an electrical guard. Runs were taken with the distance L, between G_3 and G_4, equal to 27.5 and 8.8 mm.

The electrical connections are shown in Fig. 1. G_3 and G_4 are grounded while a small sinusoidal voltage V_{ac} is applied to G_1 and G_2, which are connected together, and a dc voltage V_{dc} is applied between G_1 and the source. Thus, charged vortex rings, created near the source, acquire energy eV_{dc} as they pass through G_1. They continue to move through the field-free region and pass G_2. Because the distance between G_2 and G_3 is very small compared to the distance vortex rings travel in one period of the highest experimentally applied frequency, the electric field between G_2 and G_3 can be considered constant and vortices become velocity modulated as a result of their energy change. Between G_3 and G_4 is again a field-free region where the velocity-modulated beam propagates and then, after passing G_4, arrives finally at the collector. With the aid of a fast electrometer[4] and a signal averager, one can then easily observe the modulation in the signal current. Typically, the dc current I_{dc} is $\gtrsim 5\times10^{-13}$ A while the modulation I_{ac} (peak to

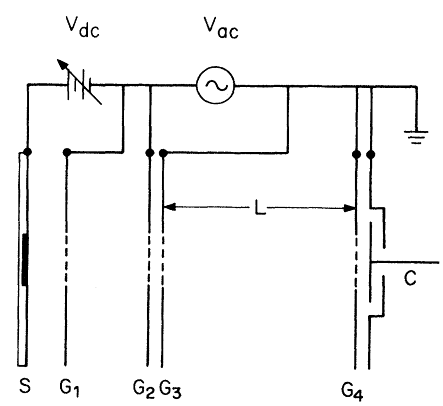

FIG. 1. Schematic diagram of the experimental cell. The cell is cylindrical in shape and is ~4 in. long and 3 in. in diameter.