Trinucleon Bound-State Properties from an Exact Solution of the Faddeev Equations with Realistic Nuclear Forces*

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The trinucleon bound-state energy, wave function, and charge form factor are calculated from a direct solution of the complete set of Faddeev equations. The nucleon-nucleon interaction (effective in the $^1S_0$ and $^3S_1$-$^3D_1$ states) is given by the Reid soft-core potential. We find $E_B = 6.7$ MeV, $P(S) = 89.7\%$, $P(S') = 1.7\%$, and $P(D) = 8.6\%$. $|F_{ch}^{^3He}(Q^2)|$ (calculated) is similar to the approximate result of Tjon, Gibson, and O'Connell, and has a minimum at $Q^2 = 15.5$ fm^{-2}.

In this paper, we report on an essentially exact solution of the complete set of homogeneous Faddeev equations for trinucleon systems. The nucleon-nucleon interaction is assumed to be effective in the $^1S_0$ and $^3S_1$-$^3D_1$ states where it is given by the Reid soft-core potential.[1] The comparison of our results with those coming from approximate solutions of the Faddeev equations or from variational calculations should give some indication of the reliability of various approximation schemes currently in use.

The trinucleon problem provides a significant means for discriminating among various so-called "realistic" nucleon-nucleon interactions, i.e., those with local one-pion–exchange tails and semiphenomenological intermediate and short-range behaviors which are adjusted to give a good fit to the properties of the deuteron and experimental nucleon-nucleon phase shifts up to around 300-MeV lab energy. However, in order to avoid misleading implications, it is essential to calculate very accurately the trinucleon properties which follow from assumed nuclear interactions. This is strikingly illustrated by the disagreement between calculations for the ^{3}He charge form factor $F_{ch}^{^3He}(Q^2)$ by Tjon, Gibson, and O'Connell[2] and by Yang and Jackson.[3] Both of these groups used the truncated Reid potential mentioned above. Yang and Jackson also used the $^1D_2$, $^3D_2$, and $^3D_3$ Reid interactions, but these are expected to have a very small effect on their results. Tjon, Gibson, and O'Connell[3] calculated the ^{3}He wave function from the Faddeev equations[4] which were simplified by neglecting the components of the Faddeev amplitude in which a nucleon is in a D state relative to the remaining nucleon pair.[5] They found $|F_{ch}^{^3He}(Q^2)|$ to have a minimum at a momentum transfer squared $Q^2 \approx 17$ fm^{-2}. Experimentally[6] $(Q^2)_{min} \approx 11.8$ fm^{-2}. Yang and Jackson[3] used a variational calculation[7] with harmonic-oscillator basis states to obtain the ^{3}He wave function, and found $(Q^2)_{min} \approx 13.5$ fm^{-2}.

During the past several years, we have set up a program[8-14] for calculating accurate solutions of the complete set of trinucleon Faddeev equations and are now completing calculations of the properties of ^{3}H and ^{3}He for a number of realistic nucleon-nucleon interactions (Hamada-Johnston,[15] Reid,[1] Bressel-Kerman-Rouben,[6] and Feshbach-Lomon[17,18] as well as phase-equivalent modifications of these[19]). A comprehensive description of our analysis and results will be published elsewhere.

We use center-of-mass trinucleon $\mathcal{L}$-$\mathcal{S}$ basis states[5,11] $|pq(Ll)\mathcal{L}W_{\mathcal{S}}^{\ r}\mathcal{S}\mathcal{J}_z\rangle$; $\vec{p} = \frac{1}{2}(\vec{k}_2 - \vec{k}_3)$, $\vec{q} = (\vec{k}_2 + \vec{k}_3 - 2\vec{k}_1)/(12)^{1/2}$, where $\vec{k}_i$ is the momentum of nucleon i, and $W_{\mathcal{S}}^{\ r}$ is a spin-isospin state with total spin $\mathcal{S} = \frac{1}{2}$ or $\frac{3}{2}$ and total isospin $\mathcal{T} = |\mathcal{T}_z| = \frac{1}{2}$. $r = A, S, +, -$ denotes, respectively, complete antisymmetry, complete symmetry, mixed overall symmetry with symmetry under 23 exchange, and mixed overall symmetry with antisymmetry under 23 exchange. The basis states are normalized so that $\langle p'q'\alpha'|pq\alpha\rangle = \delta(p'-p)\delta(q'-q)\delta_{\alpha'\alpha}/p^2q^2$. The independent components of the Faddeev amplitude for even-parity trinucleon states are listed in Table I. The bound-state wave function

TABLE I. Independent components of the Faddeev amplitude for the case of nucleon-nucleon interactions in the $^1S_0$ and $^3S_1$-$^3D_1$ states.

Component	L	l	$\mathcal{L}$	$\mathcal{S}$	r
1	0	0	0	$\frac{1}{2}$	A
2	0	0	0	$\frac{1}{2}$	–
3	2	0	2	$\frac{3}{2}$	–
4	0	2	2	$\frac{3}{2}$	–
5	2	2	2	$\frac{3}{2}$	–

is calculated from these components of the homogeneous solution of the Faddeev equations.[5,11,20]

The homogeneous solution of the five coupled two-dimensional Faddeev integral equations was obtained by using the iteration method of Malfliet and Tjon.[5,21] The bound-state energy E_B is determined as that value of the energy parameter for which the ratio of successive iterations of the Faddeev amplitudes goes to unity as the order of iteration goes to infinity. About twelve iterations were required to give satisfactory convergence. Ten- and sixteen-point Gaussian quadratures were used to do the q and p integrations, respectively. The integrals were cut off for $q > 1.7$ fm^{-1} and $p > 13$ fm^{-1}. The wave function, however, was calculated for p and q values up to 13 and 8 fm^{-1}, respectively. The Faddeev amplitudes were evaluated at fixed p, q points, and fifth-degree Lagrangian interpolation was used to evaluate the integrands at the Gaussian pivotal points required by the variable p integration limits. The results were found to be stable with respect to variations in the upper integration limits and the orders of the Gaussian quadratures.

If the Faddeev amplitudes 3, 4, and 5 in Table I are neglected, we find $E_B = -6.7$ MeV with about a 1% uncertainty. Tjon, Gibson, and O'Connell[2] find $E_B = -6.8 \pm 0.5$ MeV for this case using the iteration method. With only components 4 and 5 neglected, we find $E_B = -6.4$ MeV. Neglecting the same Faddeev components, Malfliet and Tjon[5] obtain $E_B = -6.4 \pm 0.5$ MeV and Bhatt, Levinger, and Harms[22] obtain $E_B = -6.8$ MeV using the unitary-pole approximation[23-26] (UPA) for the nucleon-nucleon t matrix. If we include components 1 through 4 or all five components, we find practically the same result, $E_B = -6.7$ MeV. Bhatt, Levinger, and Harms[22] find $E_B = -7.58$ MeV using the UPA and solving the complete set of Faddeev equations. The comparison of our calculated E_B with theirs suggests that substantial perturbation corrections must be added to UPA results. Hadjimichael and Jackson[3] find $E_B > -5.85$ and estimate an extrapolated value of $E_B = -6.25 \pm 0.25$ MeV.

In Table II, we give the components (with probabilities greater than 0.25%) of the trinucleon wave function extracted from the exact five-component Faddeev amplitude. The consistency of our numerical procedures was checked by comparing the probabilities of the + and − mixed-symmetry components. They were found to be equal (as is required by the total antisymmetry of the wave function[14]) to within an absolute difference of 0.02%.

Our values 89.7, 1.68, and 8.56% for $P(S)$, $P(S')$, and $P(D)$, respectively, are compatible with those found by Malfliet and Tjon[5] (89.9, 1.8, and 8.1%), the difference between them being due to the additional D-state components in our calculation.

TABLE II. Components of the trinucleon bound-state wave function with probabilities greater than 0.25%. $P(S) = P(^2S_{1/2}, A) = 89.75\%$; $P(S') = 2P(^2S_{1/2}, +) = 2P(^2S_{1/2}, -) = 1.68\%$; $P(D) = 2P(^4D_{1/2}, +) = 2P(^4D_{1/2}, -) = 8.56\%$.

Component	L	l	$\mathcal{L}$	$\mathcal{S}$	r	Probability (%)
1	0	0	0	$\frac{1}{2}$	A	88.3
2	2	2	0	$\frac{1}{2}$	A	1.4
3	0	0	0	$\frac{1}{2}$	−	0.8
4	1	1	0	$\frac{1}{2}$	+	0.8
5	2	0	2	$\frac{3}{2}$	−	3.1
6	0	2	2	$\frac{3}{2}$	−	1.0
7	1	1	2	$\frac{3}{2}$	+	2.5
8	3	1	2	$\frac{3}{2}$	+	1.0
9	1	3	2	$\frac{3}{2}$	+	0.3

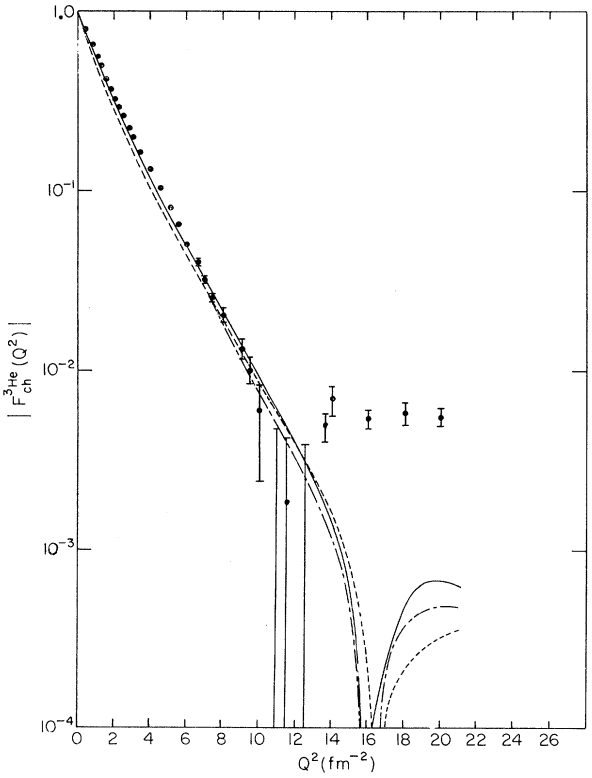

FIG. 1. ^{3}He charge form factor. Circles, data from Ref. 6. Solid, dot-dashed, and dashed curves were obtained by using the respective wave-function components 1–9; 1, 3, 4; and 1–4 in Table II.

However, our values deviate more drastically from the variational results[3] (90.56, 0.52, and 8.92%) particularly with respect to $P(S')$. This discrepancy is puzzling and will hopefully disappear when additional terms are added to the variational wave function. It is interesting to note the similarity of our results with those obtained by Delves and Hennell[27] (89.2, 1.8, and 9%) from an elaborate variational calculation using the (hard core) Hamada-Johnston potential.[15]

The calculated $|F_{ch}{}^{3He}(Q^2)|$ is plotted in Fig. 1 along with the experimental values from the work of McCarthy *et al.*[6] The analytic forms of Janssens *et al.*[28] were used for the nucleon charge form factors $f_{ch}{}^{p,n}$. Because of the large number of terms in the wave function, it was convenient to use the following general expression for the form factor:

$$2F_{ch}{}^{3He}(Q^2)=\int_0^\infty p^2\,dp\int_0^\infty q^2\,dq\sum_{L,l}\sum_{\mathcal{L},\mathcal{S}}\sum_{r',r}\sum_{\lambda=0}^{l}(-1)^\lambda\left(\frac{Q}{\sqrt{3}}\right)^\lambda q^{l-\lambda}\frac{l!}{(l-\lambda)!\lambda!}\frac{1}{2}\int_{-1}^{1}dz\,\frac{P_\lambda(z)}{q_1{}^l}$$

$$\times\langle\psi_B{}^{3He}|\,pq_1(Ll)\mathcal{L}W_{\mathcal{S}}{}^{r'}\mathcal{J}\mathcal{J}_z\rangle\langle W_{\mathcal{S}}{}^{r'}|\sum_{i=1}^{3}[f_{ch}{}^p(Q^2)\tfrac{1}{2}(1+\tau_{iz})+f_{ch}{}^n(Q^2)\tfrac{1}{2}(1-\tau_{iz})]\,|W_{\mathcal{S}}{}^r\rangle$$

$$\times\langle pq(Ll)\mathcal{L}W_{\mathcal{S}}{}^r\mathcal{J}\mathcal{J}_z|\psi_B{}^{3He}\rangle, \qquad (1a)$$

and

$$q_1=\left(q^2+\frac{Q^2}{3}-\frac{2q}{\sqrt{3}}Qz\right)^{1/2}, \qquad (1b)$$

$$\mathcal{J}=\mathcal{J}_z=\mathcal{T}=\mathcal{T}_z=\tfrac{1}{2}. \qquad (1c)$$

The dot-dashed curve was calculated using wave-function components 1, 3, and 4 of Table II. The dashed curve is calculated using components 1 through 4, and the solid curve is calculated using all of the listed components. The dot-dashed curve was calculated with the same class of wave-function components that were used in Ref. 2 and agrees quite well with the results therein. The minimum occurs at $Q^2\approx 16.2$ fm^{-2} and the ^{3}He charge radius is 2.00 fm. The "exact" (solid) curve is practically the same as the other two, with a minimum at $Q^2\approx 15.5$ fm^{-2} and charge radius 1.96 fm. The experimental ^{3}He charge radius is[6] 1.88 ± 0.05 fm. As is the case for $P(S')$ our form-factor results for $Q^2\gtrsim 10$ fm^{-2} deviate substantially from the variational results.[3]

Our results suggest that we can calculate E_B to within 0.3 MeV, the charge radius to within 0.04 fm, and the charge form factor for $Q^2\lesssim 16$ fm^{-2}, by solving the Faddeev equations retaining only components 1 and 2 in Table I. This is of considerable practical importance since it allows one to survey the implications of various phase-equivalent modifications of the nucleon-nucleon interaction without having to solve the complete set of Faddeev equations.

We are now doing calculations with phase-equivalent modifications[19] of the Reid potential, which negligibly affect the one-pion–exchange tail for nucleon separations greater than 1.0 fm. Our preliminary results indicate changes of $\lesssim 0.5$ MeV in E_B and no substantial improvement in fitting the experimental $|F_{ch}{}^{3He}|$ for $Q^2\gtrsim 10$ fm^{-2}.

A nucleon-nucleon interaction with a larger core radius (such as 0.72 fm in the case of the boundary-condition model[18]) is probably needed to bring the calculated form factor reasonably close to the experimental one.[29]

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[29]The good fit to $|F_{ch}{}^{\overline{3}He}|$ between $Q^2 = 10$ and 20 fm^{-2} obtained with a pure boundary condition model [Kim and Tubis, Ref. 13] gives some support for this conjecture.

Asymmetric Fission in the Two-Center Model

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Four-dimensional potential energy surfaces have been calculated in the asymmetric two-center model for ^{252}Fm, ^{258}Fm, ^{264}Fm, and ^{236}U. Symmetric fission is found to be preferred in ^{258}Fm, consistent with a recent observation; symmetric mass division is strongly preferred in ^{264}Fm. Asymmetric fission is preferred in ^{252}Fm, and in ^{236}U for which the fission path is investigated in more detail. The development of asymmetry in the fission of ^{236}U is described.

One of the most interesting results of calculations of collective potential-energy surfaces for heavy nuclei has been the instability of the second barrier against asymmetric deformations.[1-3] The result suggests that the mass asymmetry observed in low-energy nuclear fission of heavy elements may be due to static potential properties associated with the deformation behavior of these nuclei. This suggestion is not new, of course, having been indicated by numerous experimental evidences of fragment-structure influences in fission.[4]

The two-center model provides a method for calculating the energy levels and total potential energy of a deformed compound nucleus from its ground state to scission. It has been found in earlier two-center-model calculations,[5,6] carried out with reflection-symmetric shapes, that a level structure similar to that of the final fragments occurs quite early in the fission process,