3rd ed.
[5]J. A. Jahnke, L. Meyer, and S. Rice, Phys. Rev. A 3, 734 (1971).
[6]S. Förster, Cryogenics 3, 176 (1963).
[7]L. Bruschi and M. Santini, Rev. Sci. Instrum. 41, 102 (1970).
[8]β source target TR T2, The Radiochemical Centre, Amersham, Buckinghamshire, England.
[9]*Properties of Materials at Low Temperatures*, edited by V. J. Johnson (Pergamon, New York, 1961), Phase 1, Part 1, p. 6003.
[10]Neon N40, 99 999% quoted purity, from Societã Italiana Ossigeno, Milan, Italy.
[11]S. Cunsolo, Nuovo Cimento 21, 76 (1961).
[12]H. T. Davis, S. A. Rice, and L. Meyer, J. Chem. Phys. 37, 947 (1962).

Mechanism for Anomalous Current Penetration*

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A localized collisionless instability that occurs in a nonuniform or skin current flowing along an external magnetic field is shown to produce an anomalous viscosity giving cross-field diffusion of the current. The instability can occur under conditions where the ion-sound wave is stable and is then the dominant process for penetration.

In this Letter we present a collisionless mechanism for the penetration of a skin current into a plasma under conditions where the ion-sound modes are stable. The collisionless penetration is shown to occur through turbulent $\vec{E}\times\vec{B}$ drifts of the electrons produced by fluid instabilities occurring in the nonuniform current profile. Previous studies[1-3] have shown that shear in a beam or current profile can drive fluid instabilities. The growth rates obtained[1,2] are sensitive functions of the angle of wave propagation to the magnetic field, and the angle remains an arbitrary parameter. The theory presented here gives a self-consistent determination of the angle of propagation and yields a growth rate and critical gradient free of the angle parameter.

We show that a nonuniform current $j_z(x)$ flowing along a magnetic field $B_0\hat{e}_z$ can diffuse across the magnetic field by turbulence from a fluid instability with a maximum growth rate of the order $\gamma \sim \frac{1}{2}|du/dx|$, where u is the drift velocity of the electrons in the current $j_z = -en_0u$. The unstable mode has a small parallel electric field which is determined self-consistently in the presence of the sheared magnetic field produced by the current j_z. In the following analysis the plasma density and temperature are assumed sufficiently uniform with respect to the current nonuniformity to neglect their effect. The important electron-temperature gradient modes[3] as well as the ion-sound instability are driven by resonant electrons, in contrast to the present fluid instability driven by $|du/dx|$. As a consequence, the quasilinear change in the parallel electron velocity distribution,[4] which can saturate the resonant temperature gradient and ion-sound instabilities, has little effect on the present instability. Instead, we estimate that saturation of the current gradient mode occurs through the anomalous viscosity and diffusion of the current profile in x space. The anomalous viscosity in the saturated state is estimated in Eq. (17) below.

The equilibrium plasma in slab geometry has a nonuniform current $j_z(x)$ which can be taken as a linear gradient in the neighborhood of the localized mode with $j_z(x) = -en_0(u + x\,du/dx)$. The shear in the magnetic field $\vec{B} = B_y(x)\hat{e}_y + B_0\hat{e}_z$ is given by

$$dB_y/dx = -4\pi e n_0 u/c$$

which yields $B_y(x) = B_y(0) + B_0(x/L_s)$, where the distance L_s given by $L_s^{-1} = \omega_{pe}^2(u/c^2)\omega_{ce}$ is the characteristic distance over which the magnetic field rotates through an appreciable angle.

We take as our basic equations the fluid equations with an electrostatic field $\vec{E} = -\nabla\varphi(\vec{x}, t)$. For the electrons we have

$$\frac{\partial n_e}{\partial t} + \nabla\cdot\left(\frac{cn_e\vec{B}\times\nabla\varphi}{B^2}\right) + \frac{\partial}{\partial x_\parallel}(n_e u) = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{c\vec{B}\times\nabla\varphi}{B^2}\cdot\nabla u + u\frac{\partial u}{\partial x_\parallel} = \frac{e}{m_e}\frac{\partial\varphi}{\partial x_\parallel}, \tag{2}$$

$$\epsilon_\perp\nabla_\perp^2\varphi = -4\pi e(n_i - n_e), \tag{3}$$

where $\epsilon_\perp = 1 + \omega_{pe}^2/\omega_{ce}^2$, and it is assumed that $k_\perp v_e < \omega_{ce}$, $\omega_{ce} > \omega > k_\parallel v_e$, where $v_{e,i} = (T_{e,i}/m_{e,i})^{1/2}$.

The mode is basically an electron oscillation, and qualitatively the ion motion can be entirely neglected. The ion motion does contribute in general, however, and a simple regime occurs for relatively hot ions where $k_\perp v_i > \gamma > \omega_{ci}/\pi$. In this case the ions are essentially unmagnetized in the wave and contribute $4\pi\rho_i = (\omega_{pi}^2/v_i^2)\varphi$ to the charge density. For beams where $\gamma \simeq \frac{1}{2}u' \sim u\omega_{pe}/2c$, the ion motion can be neglected entirely, and the corresponding results are obtained from the following in the infinite ion mass limit with constant v_i.

Linearizing Eqs. (1), (2), and (3) and taking the perturbation to vary as $\varphi(x)\exp[i(k_y y + k_z z - \omega t)]$, we obtain the equation for the potential,

$$\epsilon_\perp \frac{d^2\varphi}{dx^2} + \left[-\epsilon_\perp k_y^2 - \frac{\omega_{pi}^2}{v_i^2} + \omega_{pe}^2 k_\parallel \left(k_\parallel - \frac{k_y}{\omega_{ce}} \frac{du}{dx} \right) (\omega - k_\parallel u)^{-2} \right] \varphi = 0, \tag{4}$$

where $k_\parallel(x) = [k_y B_y(x) + k_z B_0]/B$, and the Doppler shift is negligible since $\omega > k_\parallel v_e > k_\parallel u$.

The new feature of Eq. (4), giving rise to a localized instability, is the splitting of the usual $k_\parallel^2 = 0$ turning points by the sheared current du/dx. The potential in Eq. (4) forms a well for $(\omega - k_\parallel u)^2 < 0$ and $0 < k_\parallel < k_y u'/\omega_{ce}$. Choosing either k_z or the arbitrary origin $x = 0$ such that $k_y B_y(0) + k_z B_0 = 0$ and letting $\omega = i\gamma$, we obtain from Eq. (4)

$$\epsilon_\perp \frac{d^2\varphi}{dx^2} + \left\{ -\epsilon_\perp k_y^2 - \frac{\omega_{pi}^2}{v_i^2} + \frac{\omega_{pe}^2 k_y^2}{\gamma^2} \left[\left(\frac{u'}{2\omega_{ce}} \right)^2 - \frac{(x - x_0)^2}{L_s^2} \right] \right\} \varphi = 0, \tag{5}$$

where $x_0/L_s = u'/2\omega_{ce}$. The fastest growing mode from Eq. (5) is

$$\varphi(x) = \exp[-(\omega_{pe} k_y / 2\gamma \epsilon_\perp^{1/2} L_s)(x - x_0)^2], \tag{6}$$

and the quantization condition determining the growth rate is

$$\frac{\omega_{pe} k_y}{\gamma \epsilon_\perp^{1/2} L_s} + k_y^2 + \frac{1}{\epsilon_\perp \lambda_{Di}^2} - \frac{\omega_{pe}^2 k_y^2}{\gamma^2 \epsilon_\perp} \left(\frac{u'}{2\omega_{ce}} \right)^2 = 0. \tag{7}$$

Solving Eq. (7) for the growing mode we have

$$\gamma = \frac{\omega_{pe} k_y \lambda_{Di}^2}{2\epsilon_\perp^{1/2} L_s} \left\{ \left[1 + \frac{T_e}{T_i} \frac{c^2}{v_e^2} \left(\frac{c}{\omega_{pe}} \frac{u'}{u} \right)^2 (k_y^2 \lambda_{Di}^2 + \epsilon_\perp^{-1}) \right]^{1/2} - 1 \right\} (k_y^2 \lambda_{Di}^2 + \epsilon_\perp^{-1})^{-1}. \tag{8}$$

The growth rate increases linearly with k_y up to $k_y \lambda_{Di} \simeq \epsilon_\perp^{-1/2}$, where the maximum growth rate

$$\gamma_{\max} \simeq \tfrac{1}{2} |du/dx| \, \omega_{pe}/\omega_{ce} \epsilon_\perp^{1/2}$$

is reached. The condition on the current gradient required for instability follows from requiring consistency with the assumption $\gamma > \bar{k}_\parallel v_e$, where $\bar{k}_\parallel = k_y u'/2\omega_{ce}$ is the average $k_\parallel(x)$ in the normal mode. (At smaller growth rates a kinetic theory description is necessary.) At small current gradients the growth rate given by Eq. (8) reduces to

$$\gamma \simeq (k_y c^2 / 4\omega_{pe} \omega_{ce} \epsilon_\perp^{1/2})(u')^2/|u|. \tag{9}$$

Imposing the condition $\gamma > 2\bar{k}_\parallel v_e$, we obtain the critical current gradient

$$\frac{c}{\omega_{pe}} \left| \frac{1}{u} \frac{du}{dx} \right| > 4 \frac{v_e}{c} \epsilon_\perp^{1/2} \tag{10}$$

required for fluid instability. Additional requirements on the plasma gradients are readily obtained from the above for consistent neglect of the ion-cyclotron resonances ($\pi\gamma/\omega_{ci} > 1$) and the temperature and density gradients ($\gamma > \omega_{T,n}^*$). Further study of instability including these effects will be presented in a subsequent paper.

For current gradients appreciably greater than the critical gradient we obtain from Eq. (8) the growth rate

$$\gamma \simeq \frac{\omega_{pe}}{2\omega_{ce} \epsilon_\perp^{1/2}} \left| \frac{du}{dx} \right| \frac{k_y \lambda_{Di}}{(k_y^2 \lambda_{Di}^2 + \epsilon_\perp^{-1})^{1/2}}. \tag{11}$$

The width of the normal mode centered about $k_\parallel(x_0) = k_y u'/2\omega_{ce}$ is related to the growth rate by Eq. (6)

and is

$$\langle(\Delta x)^2\rangle = \frac{1}{2}\frac{c^2}{\omega_{pe}^{\;2}}\left|\frac{\lambda_{\mathrm{D}i}}{u}\frac{du}{dx}\right|(k_y^{\;2}\lambda_{\mathrm{D}i}^{\;2}+\epsilon_\perp^{\;-1})^{-1/2}. \tag{12}$$

For a skin current with a gradient exceeding the critical gradient, a spectrum of unstable modes will develop and produce diffusion of the background current profile. Since the parallel electric field in the waves is small, the nonlinear effects of electron trapping are unimportant compared to the convective nonlinearities contained in Eq. (2). Likewise, diffusion of the parallel electron distribution function is not a significant stabilizing effect for these fluid instabilities. For a spectrum of unstable waves, we have

$$\varphi(\vec{x},t) = \textstyle\sum_{k_y}\varphi_{k_y}(x,t)\cos(k_y y+\delta_k), \tag{13}$$

and from Eq. (2)

$$u(\vec{x},t) = -\frac{e}{m_e}\left(\frac{x}{L_s}-\frac{u'}{\omega_{ce}}\right)\sum_{k_y}\frac{k_y\varphi_{k_y}(x,t)}{\gamma_{k_y}}\sin(k_y y+\delta_k), \tag{14}$$

where $\varphi_{k_y}(x,t)=\varphi_{k_y}(x)\exp(\gamma_{k_y}t)$. Substituting Eqs. (13) and (14) into Eq. (2) and averaging over the wave spectrum, we obtain

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(\mu_A\frac{\partial u(x,t)}{\partial x}\right), \tag{15}$$

where

$$\mu_A = \sum_{k_y}\frac{c^2 k_y^{\;2}|\varphi_{k_y}(x,t)|^2}{2B^2\gamma_{k_y}} = \frac{\tilde{v}_x^{\;2}}{\bar{\gamma}}. \tag{16}$$

The turbulent viscosity μ_A produced by the short-wavelength modes $k_y\lambda_{\mathrm{D}i}\approx\epsilon_\perp^{\;-1/2}$ has a weak stabilizing effect on the long-wavelength modes since $\gamma > k_\perp^{\;2}\mu_A$. On the other hand, the short-wavelength modes become strongly nonlinear and presumably saturate when, in one growth period for the wave, the convective velocity $\bar{v}_x = c\bar{E}_y/B$ is sufficiently large to take the electrons across the entire width of the mode. Taking this limit ($\bar{v}_x\sqrt{\gamma} = \Delta x$) for the amplitude of the turbulence, we obtain

$$\mu_A = \frac{1}{8}\frac{c^2}{\omega_{ce}\omega_{pe}}\frac{\lambda_{\mathrm{D}i}}{|u|}\left|\frac{du}{dx}\right|^2 \tag{17}$$

as an estimate of the anomalous viscosity. [In the high-frequency limit where the ion mass is taken to be infinite the result is $\mu_A = \frac{1}{8}(c^4/\omega_{pe}^{\;3}\omega_{ce}\epsilon_\perp^{\;1/2}) \times u^{-2}|du/dx|^3$.] The level of the fluctuating potential in Eq. (16) in the saturated state corresponding to Eq. (17) is of the order $e\bar{\varphi}/T \simeq (cu'/\omega_{pe}u)^{3/2} \times (u^2/cv_e)^{1/2} \ll 1$. We note that the diffusion equation (15) for the profile is nonlinear with $\mu_A \propto (u')^2/u$, and as a consequence the diffusion rate is considerably enhanced in profiles with large gradients.

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[1]A. B. Mikhailovskii and A. A. Rukhadze, Zh. Tekh. Fiz. 35, 2143 (1965) [Sov. Phys. Tech. Phys. 10, 1644 (1966)].

[2]A. Hirose and I. Alexeff, Phys. Rev. Lett. 26, 949 (1971).

[3]C. S. Liu, Phys. Rev. Lett. 27, 1637 (1971).

[4]R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory* (Benjamin, New York, 1969), pp. 81–84.