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Nonperturbative Evaluation of the Anomalies in Low-Energy Theorems*

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It is shown that the anomalous constant S associated with the soft $\pi^0 \to 2\gamma$ decay amplitude is determined by a product of high-energy electroproduction and annihilation cross sections. A similar result holds for the coupling of the trace of the stress-energy tensor to photons. Wilson's theory of broken scale invariance is assumed. We observe that the Fritzsch–Gell-Mann algebra of bilocal operators uniquely specifies disconnected contributions, and, in particular, implies $S=\frac{1}{6}$, the quark field-theoretic value.

Several authors[1] have observed that gauge and chiral invariance determine all partially conserved axial-vector current (PCAC) anomalies up to an unknown multiplicative constant, the anomaly[2] associated with $\pi^0 \to 2\gamma$ decay. This constant has been extensively studied[3] in perturbation theory, where spinor triangle diagrams are responsible for its existence.

In this note, it will be shown that the theory of broken scale invariance[4] directly relates high-energy cross sections to anomalies in low-energy theorems. Both anomalous PCAC and the analogous calculation for the hadronic stress-energy tensor $\theta_{\mu\nu}(x)$ can be treated in this nonpertubative fashion. Instead of having to introduce the couplings of elementary fermions to currents, we shall consider constants such as the ratio of cross sections for high-energy e^+e^- annihilation,

$$R=\sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-). \tag{1}$$

The derivation makes use of algebraic constraints which relate short-distance expansions for products of several operators to those of less complicated products.

Suppose that products of local operators $A(x), B(x), C(x), \cdots$ are expanded at short distances in terms of c numbers $\mathcal{C}_n(x), f_n(x,y), \cdots$, and operators $O_n(x)$ [4,5]:

$$T\{A(x)B(0)C(y)\} = \sum_n f_n(x,y) O_n(0).$$

Each c-number function $f_n(x,y)$ contains poles and branch cuts in the variables $u=x^2-i\epsilon$, $v=y^2-i\epsilon$, and $w=(x-y)^2-i\epsilon$. Consider coefficients of the u singularities. Because the expansion

$$T\{A(x)B(0)\} = \sum_m \mathcal{C}_m(x) O_m'(0)$$

converges weakly, the formula

$$T\{A(x)B(0)C(y)\} = \sum_m \mathcal{C}_m(x) T\{O_m'(0)C(y)\}$$

holds only if y remains independent of the limit $x\to 0$ (i.e., $x \ll y$). However, this does not prevent y from being in the short-distance region, where the expansion

$$T\{O_m'(0)C(y)\} = \sum_n \mathcal{C}_{mn}(y) O_n(0)$$

is valid. Consequently, the u singularities of $f_n(x,y)$ must satisfy the consistency condition[6]

$$f_n(x,y) \underset{x\ll y}{\to} \sum_m \mathcal{C}_m(x)\mathcal{C}_{mn}(y). \qquad (2)$$

The discussion is easily generalized to include products of several operators and limits $x_i, \cdots, x_j \ll x_k, \cdots, x_l$ of the corresponding coordinate differences. Each distinct limit[7] (or succession of limits) leads to a constraint equation. To be consistent, a current algebra which is postulated to be valid at short distances or on the light cone must obey these algebraic conditions.

The application of Eq. (2) to the anomalous low-energy theorem for $\pi^0 \to 2\gamma$ decay will now be described.

In terms of $J_\mu(x)$, the electromagnetic current for hadrons, and $J_\mu{}^5(x)$, the third isospin component of the axial-vector current, the definition of Adler's anomalous constant[2,3] S is

$$S = -\tfrac{1}{12}\pi^2\epsilon^{\mu\nu\alpha\beta}\iint d^4x\, d^4y\, x_\mu y_\nu T\langle 0|J_\alpha(x)J_\beta(0)\partial^\gamma J_\gamma{}^5(y)|0\rangle. \qquad (3)$$

PCAC and the experimental value for the $\pi^0 \to 2\gamma$ decay amplitude imply[3,8] $S \simeq +0.5$. Wilson[4] has shown that S is completely determined by the leading scale-invariant singularity $G_{\alpha\beta\gamma}$ of the short-distance expansion

$$T_{\alpha\beta\gamma}(x,y) = T\{J_\alpha(x)J_\beta(0)J_\gamma{}^5(y)\} = G_{\alpha\beta\gamma}(x,y)I + \cdots. \qquad (4)$$

The theory of broken scale invariance requires $G_{\alpha\beta\gamma}$ to be a homogeneous function of degree -9 in x and y.

If no further restrictions were implied by the theory, computing $G_{\alpha\beta\gamma}(x,y)$ would be a hopelessly complex problem in strong-interaction dynamics. However, Schreier[9] has observed that, in a strictly conformal-invariant world, $G_{\alpha\beta\gamma}$ must be proportional to

$$\Delta_{\alpha\beta\gamma}(x,y) = \tfrac{1}{4}i\,\mathrm{Tr}\gamma_\alpha\gamma\cdot x\gamma_\beta\gamma\cdot y\gamma_\gamma(\gamma\cdot x - \gamma\cdot y)\gamma_5(uvw)^{-2}. \qquad (5)$$

(The γ-matrix notation is a convenient shorthand; no dependence on perturbation theory is implied.) For the real world, it can be shown[10] that this theorem is implied by $\partial^\alpha J_\alpha = 0$ and the dimensional constraints[4]

$$\dim\partial^\gamma J_\gamma{}^5 < 4, \quad \dim\theta_\mu{}^\mu < 4, \qquad (6)$$

and is not affected by the failure of the vacuum to preserve scale and conformal invariance.

Now consider the limit $x \ll y$ of Eq. (4). A consistency condition for $G_{\alpha\beta\gamma}$ will be deduced from the expansions

$$T\{J_\alpha(x)J_\beta(0)\} = R(g_{\alpha\beta}x^2 - 2x_\alpha x_\beta)I/(\pi u)^4 + K\epsilon_{\alpha\beta\lambda\mu}x^\lambda J^{\mu 5}(0)/3\pi^2u^2 + \cdots, \qquad (7)$$

$$T\{J_\mu{}^5(0)J_\gamma{}^5(y)\} = R'(g_{\mu\gamma}y^2 - 2y_\mu y_\gamma)I/(\pi v)^4 + \cdots. \qquad (8)$$

We assume that the dots stand for derivatives of operators, terms which break scale invariance, and operators which have different isospin, dimension, spin, or parity. The constants R [also given by Eq. (1)], K, and R' are treated as parameters to be measured experimentally; special values are considered only when making contact with previous work. Take the limit $x \ll y$ of $G_{\alpha\beta\gamma} \propto \Delta_{\alpha\beta\gamma}$, and isolate the leading contribution, which is determined by the $J^{\mu 5}(0)$ term in Eq. (7)[11]:

$$G_{\alpha\beta\gamma}(x,y) \underset{x\ll y}{\to} (K\epsilon_{\alpha\beta\lambda\mu}x^\lambda/3\pi^2u^2)[R'(\delta_\gamma{}^\mu y^2 - 2y^\mu y_\gamma)/(\pi v)^4]. \qquad (9)$$

This procedure fixes the constant of proportionality:

$$G_{\alpha\beta\gamma}(x,y) = (KR'/3\pi^6)\Delta_{\alpha\beta\gamma}(x,y). \qquad (10)$$

When written as a Ward identity, Eq. (3) becomes a surface integral in (x,y) space.[4] Given Eq. (10), the integration may be performed explicitly[10]; the result is an exact formula,

$$3S = KR', \qquad (11)$$

for the anomalous constant.[12]

Equation (11) demonstrates the intimate relation between PCAC anomalies and processes induced by highly virtual photons. The constant K is given by Bjorken's sum rule for polarized deep-inelastic electroproduction[13]; in addition, K^2 is measured by the cross section[14] for $e^+(q_+)+e^-(q_-)\to\mu^++\mu^-+\pi^0(q)$ in the limit $(q^++q^-)^2\to\infty$, $\vec{q}$ fixed. [The relevant formulas were derived assuming $K=1$, the value fixed by $U(6)\otimes U(6)$ algebra[15].] Chiral invariance at short distances implies that R' is the isovector part of the ratio R defined by Eq. (1).

Bjorken[13] has shown that the choice $K=1$ leads to large asymmetries in polarized inelastic electroproduction, so it should be possible to determine K experimentally. However, although R' is given by a well-defined cross section, it cannot be directly measured. If we allow for the possibility that $J_\mu(x)$ is not a pure SU(3) octet operator $(4R'\leq 3R)$, the bound

$$4S\leq KR \tag{12}$$

may provide a practical test of the PCAC result $S\simeq 0.5$.[16]

Clearly, this discussion is related to the construction of current algebras on the light cone. Fritzsch and Gell-Mann[17] generate an algebra of bilocal operators from the quark model, and then propose that this algebra be treated as a nonperturbative abstraction. Unfortunately, this amounts to a postulate that chiral $SU(2)\otimes SU(2)$ is a badly broken symmetry,[17] because the algebra requires $K=1$, $R'=\frac{1}{2}$, and hence $S=\frac{1}{6}$. The existence of constraints such as Eq. (2) means that neither K nor R' may be altered without changing many other coefficients in the algebra. For example, any algebra in which the product $\mathfrak{F}_\alpha{}^a(z)\mathfrak{F}_\beta{}^b(0)$ of SU(3) currents $\mathfrak{F}_\mu{}^a(x)$ $(a=1,\cdots,8)$ does not contain the "spin-0 term" $z_\alpha z_\beta z^\lambda\mathfrak{F}_\lambda{}^c(0)/z^6$ at small z must obey the constraint

$$\lim_{z\ll y}\lim_{y,z\ll x}\mathfrak{F}_\alpha{}^a(x)\mathfrak{F}_\beta{}^b(y)\mathfrak{F}_\gamma{}^c(z)\mathfrak{F}_\delta{}^d(0)\to -R'f^{bcd}z^\mu s_{\gamma\delta\mu\nu}(\delta_\beta{}^\nu y^2-2y^\nu y_\beta)\mathfrak{F}_\alpha{}^a(0)/2\pi^6z^4y^8+\cdots \tag{13}$$

at small x, with $s_{\gamma\delta\mu\nu}=g_{\gamma\mu}g_{\delta\nu}+g_{\gamma\nu}g_{\delta\mu}-g_{\gamma\delta}g_{\mu\nu}$. Therefore, even if the Fritzsch–Gell-Mann hypothesis is supposed to apply only to q-number contributions, the result $S=\frac{1}{6}$ cannot be evaded. In fact, any disconnected term in the algebra can be obtained by this procedure.

Some anomalies (such as those[1] for $\gamma\to 3\pi$, $2\gamma\to 3\pi$) may be computed without specifying the leading singularity of an operator product throughout the short-distance region. Consider the vertex[18]

$$\langle\gamma(\epsilon_1,k_1),\gamma(\epsilon_2,k_2)|\theta_\mu{}^\mu|0\rangle=(\epsilon_1\cdot\epsilon_2k_1\cdot k_2-\epsilon_1\cdot k_2\epsilon_2\cdot k_1)F((k_1+k_2)^2). \tag{14}$$

The formula analogous to Eq. (3) is

$$F(0)=-\tfrac{1}{3}\pi\alpha\iint d^4x\,d^4y\,x\cdot yT\langle 0|J^\beta(x)J_\beta(0)\theta_\mu{}^\mu(y)|0\rangle, \tag{15}$$

where α is the fine-structure constant. We want to replace $2y_\lambda\theta_\mu{}^\mu(y)$ by $\partial_y{}^\mu(2y_\lambda y^\nu-\delta_\lambda{}^\nu y^2)\theta_{\mu\nu}(y)$, integrate by parts with respect to y, and obtain equal-time commutators in the usual way. However, the x integration must first be restricted to $|x_0|\geq\eta$ (where η is fixed at a small postive value), so that the $x,y\to 0$ region of the product

$$T\{J_\alpha(x)J_\beta(0)\theta_{\mu\nu}(y)\}=K_{\alpha\beta\mu\nu}(x,y)I+\cdots \tag{16}$$

can be avoided. For example, the expansion for $T\{J_\alpha(x)\theta_{\mu\nu}(y)\}\sim(x-y)^{-4}$ may be substituted in $T\{J_\alpha(x)\times J_\beta(0)\theta_{\mu\nu}(y)\}$ (to form an equal-time commutator) only if the condition $x-y\ll\eta$ is satisfied. By performing the y integration in this fashion, we obtain the formula

$$F(0)=-\tfrac{1}{6}i\pi\alpha\int_{|x_0|\geq\eta}d^4x\,\partial_x{}^\nu\{x^2x_\nu T\langle 0|J^\alpha(x)J_\alpha(0)|0\rangle\}+O(\eta),$$

which immediately reduces to the desired result,

$$F(0)=2R\alpha/3\pi. \tag{17}$$

Although the singularity $K_{\alpha\beta\mu\nu}$ is responsible for the presence of this anomaly, its functional form is not needed; only the region $x-y\ll x,y$ contributes to the Ward identity.

If the dilation current is partially conserved,[17,19] Eq. (17) leads to the prediction ("anomalous PCDC"),

$$F_\epsilon G_{\epsilon\gamma\gamma}\approx 2R\alpha/3\pi, \tag{18}$$

for the coupling $(\epsilon_1\cdot\epsilon_2k_1\cdot k_2-\epsilon_1\cdot k_2\epsilon_2\cdot k_1)G_{\epsilon\gamma\gamma}$ of the $\epsilon(700)$ meson to photons, where $|F_\epsilon|\approx 100$ MeV is given by

$$\langle\epsilon(q)|\theta_{\mu\nu}(0)|0\rangle=\tfrac{1}{3}F_\epsilon(g_{\mu\nu}q^2-q_\mu q_\nu).$$

In perturbation theory for the Fermi-Yang model ($R=1$), Eq. (18) reduces to an old result of Schwinger's.[2]

A detailed account of the current-algebraic treatment of anomalies is being prepared.

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Note added in proof.—M. Chanowitz and J. Ellis [SLAC Report No. SLAC-PUB-1028, 1972 (unpublished)] have independently discovered Eqs. (17) and (18). Their analysis is performed in momentum space.

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