determined accurately only to the lowest order. The discovery made in the present work offers us a way to overcome this Jastrow barrier. Such a method is currently being applied to investigate properties of the weakly interacting Bose gas, and more importantly liquid helium. The results will be reported elsewhere.

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Particle Loss in a Toroidally Symmetric Cusp

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We show that previous calculations of the loss rate of ions from toroidally symmetric line cusps need substantial revision. The symmetry of the toroid produces a conserved generalized momentum which prevents a certain class of particles from penetrating the mirrorlike magnetic fields which exist above the cusp points. The importance of this reduction in loss rate is explained for several geometries including Tormac which can now have a loss rate substantially smaller than a mirror device.

Previous calculations of the loss rate of ions from toroidally symmetric cusp, magnetic containment geometries are based on the existence of a hole in physical space at the cusp line through which ions can rapidly escape.[1] A substantial correction to this rate is needed because a translationally symmetric geometry, like an infinite or toroidal line cusp, produces a conserved generalized momentum. This prevents some ions from aligning their velocity vectors closely enough with the magnetic field direction to go through the larger, mirrorlike magnetic field, which in practical systems exists above the cusp lines. Such malaligned particles cannot escape from the plasma until the conservation of momentum is broken by collisions or instabilities. Consequently, the time constant is controlled by these relatively slow processes. The conservation of a generalized momentum in such symmetric geometries has been previously discussed, but its implications have not been fully exploited[2]; probably because the time constant of a spindle cusp, which is usually treated, is not affected by the conserved generalized momentum.

To find the conserved generalized momentum we consider coordinates

$$X = (R + x)\cos\psi, \tag{1}$$

$$Z = (R + x)\sin\psi, \tag{2}$$

$$Y = y, \tag{3}$$

where X, Y, and Z are the Cartesian coordinates, R is the major radius of the toroid, the angle ψ is measured around the major circle of the system, and the cusp is in the x-y plane (see Fig. 1).

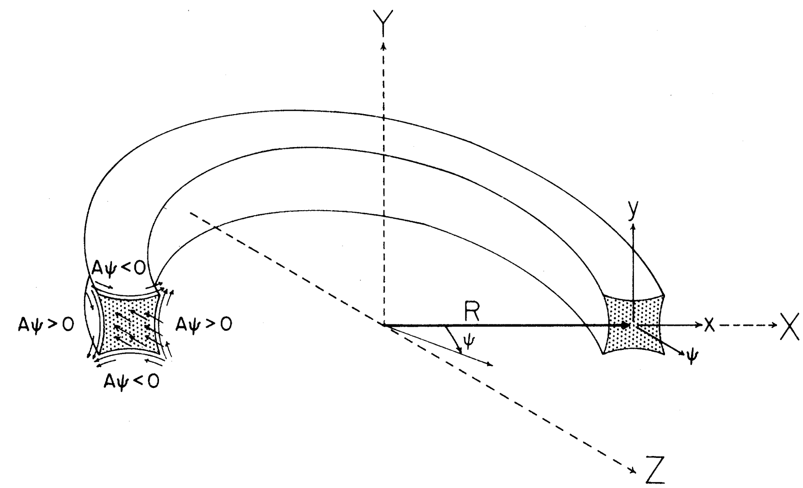

FIG. 1. Toroidally symmetric four-pole cusp. Shaded area indicates trapped plasma with totally enclosed toroidal magnetic field B_ψ.

The line element in the toroidal coordinates is

$$dr^2 = dx^2 + dy^2 + (R+x)^2 d\psi^2, \tag{4}$$

and the Lagrangian is

$$L = \tfrac{1}{2} m[\dot{x}^2 + \dot{y}^2 + (R+x)^2\dot{\psi}^2] + e\dot{x}A_x + e\dot{y}A_y + e(R+x)\dot{\psi}A_\psi, \tag{5}$$

with A_x, A_y, and A_ψ being the components of the vector potential; $\vec{B} = \nabla \times \vec{A}$.

The toroidal symmetry of the system implies that the magnetic field is independent of ψ. It is possible to construct the vector potential of such a field without any ψ dependence. The three componts of the magnetic field are given by

$$B_x(x,y) = \frac{1}{R+x}\frac{\partial(R+x)A_\psi}{\partial y} = \frac{\partial A_\psi(x,y)}{\partial y}, \tag{6}$$

$$B_y(x,y) = -\frac{1}{R+x}\frac{\partial(R+x)A_\psi(x,y)}{\partial x}, \tag{7}$$

$$B_\psi(x,y) = \frac{\partial A_x(x,y)}{\partial y} - \frac{\partial A_y(x,y)}{\partial x}. \tag{8}$$

Since the vector potential is independent of ψ, the Lagrangian is independent of ψ, and the ψ component of the generalized momentum

$$P_\psi = \partial L/\partial\dot{\psi} = m(R+x)^2\dot{\psi} + e(R+x)A_\psi(x,y) \tag{9}$$

is conserved.

In the interior of the cusp there is no magnetic field component in the x-y plane, and Eqs. (6) and (7) imply that in this region

$$A_\psi = a/(R+x), \tag{10}$$

where a is a constant. Putting this expression for the vector potential in Eq. (9), we have

$$(R+x)^2\dot{\psi} = \text{const} \tag{11}$$

in the cusp interior. If we let θ be the angle between the ψ axis and the velocity of the particle, and remember that the magnitude of the velocity is conserved, Eq. (11) implies that when a particle is in the interior of the cusp

$$\cos\theta = b/(1+x/R), \tag{12}$$

where b is a constant. We should stress that in the collisionless limit each and every time a particle leaves the plasma region to enter the sheath, or whenever it crosses a line where $A_\psi = 0$, it has the same value of b.

In the cusp geometry the plasma is bounded by surfaces of constant magnetic intensity B_0, but it is possible, and in fact usual, for the magnetic field intensity to increase along the field lines above the cusp points to some maximum value $B_{\max}$. Particles which penetrate beyond the cusp lines will be reflected by a magnetic mirror back into the cusp region if the angle its velocity makes with the magnetic field is greater than α, where

$$\cos^2\alpha = (B_{\max} - B_0)/B_{\max}. \tag{13}$$

In the usual mirror geometry a particle once reflected continues to be reflected because its motion is adiabatic. The toroidal cusp system differs in that a reflected particle may again try to penetrate the magnetic mirror after nonadiabatic motion in the sheath or the main plasma. However, the fact that the ψ component of the generalized momentum is conserved means that the particle motion, though nonadiabatic, is not arbitrary.

To explore the implications of this statement, we now consider a more specific model; although, we expect that our main result is substantially model independent. We assume that the plasma is separated from the vacuum region by a sharp boundary. In this approximation, a particle can only leave the plasma by cutting an $A_\psi = 0$ surface near the cusp line. We further assume that outside the boundary the scale of the variation of the magnetic field is much smaller than the ion gyroradius. This allows us to use the magnetic moment as a good adiabatic invariant from the instant the particle is outside the main plasma region; moreover, using the total instantaneous perpendicular velocity to evaluate the adiabatic invariant instead of averaging over a period will only lead to a small error.

In the cusp region a particle can experience a $\nabla B \times \vec{B}$ drift. We assume that this drift velocity is small enough to be ignored in the numerical evaluation of the adiabatic invariant. On the other hand, with a B_ψ present the drift is such that upon reflection from the high-B region the

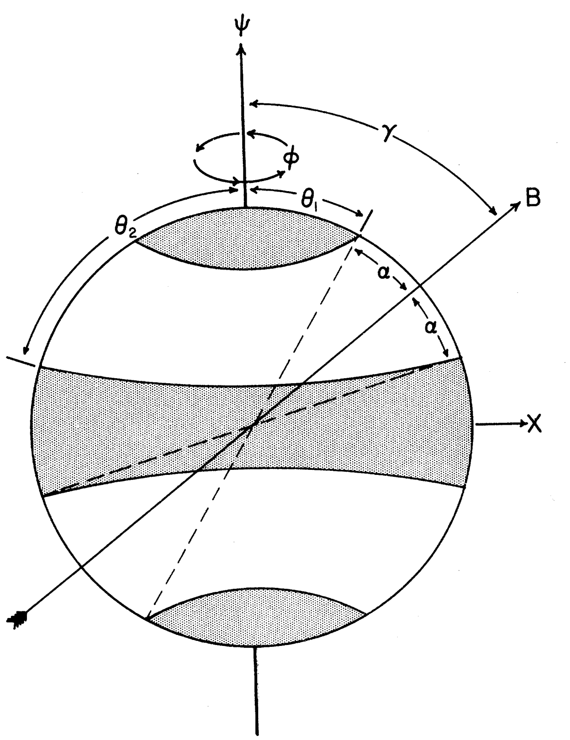

FIG. 2. Diagram of velocity space attached to a point on a boundary magnetic field line. Shaded area indicates trapped region.

particle may drift away from the $A_\psi = 0$ line and not re-enter the main plasma at the same point it left. In this case, however, the particle will pass along the sheath and re-enter the plasma through a neighboring cusp. This can be seen if we assume the particle motion is adiabatic during the entire excursion outside of the plasma and along the sheath. Than, in the guiding-center approximation, the $\nabla B \times \vec{B}$ drift away from the $A_\psi = 0$ surface at one cusp is balanced by a $\nabla B \times \vec{B}$ drift toward the $A_\psi = 0$ surface at the second cusp.

Thus, to escape, the particle must move along an open field line with its velocity within the angle α of the magnetic field [Eq. (13)]. It is easy to see that this implies that θ_c, the angle between the particle's velocity and the ψ axis when it crosses into the adiabatic region, must be in the range[3]

$$\begin{aligned} &\gamma - \alpha \le \theta_c \le \gamma + \alpha, \\ &\pi - (\gamma + \alpha) \le \theta_c \le \pi - (\gamma - \alpha), \end{aligned} \tag{14}$$

where γ is the angle the B field makes with the ψ axis at the cusp[4] (see Fig. 2). Since θ_c has the same value as θ in the adjacent interior region of the cusp (A_ψ is continuous at the boundary), Eq. (13) tells us that only certain particles can ever satisfy relation (14); the rest must remain trapped in the plasma region until they suffer a collision or an instability develops.

If we let P be the fraction of the sphere of possible velocity directions which contains particles that cannot escape, and let D be the distance between the furthest cusp points in the x direction, then from Fig. 3 we can see that[3]

$$P = 1 - \frac{\cos\theta_1}{1 - D/R} + \frac{\cos\theta_2}{1 + D/R}, \tag{15}$$

with $\theta_1 = \gamma - \alpha$ and $\theta_2 = \gamma + \alpha$. In Fig. 3 we have plotted curves of constant P for the case in which D/R is negligibly small.

In order to apply these results we must note that there are two time constants associated with a cusp geometry. One time constant, τ_g, is derived by using the model of Grad[1] as refined by Grossman[5] and applies to the region of phase space where particles can escape the cusp without a prior particle collision. The other time constant, τ_c, is applicable to that portion of phase space where particles are trapped and depends on the volume in phase space and the collision frequency. An approximate expression is

$$\tau_c \simeq \tau_{ii}(1 - P), \tag{16}$$

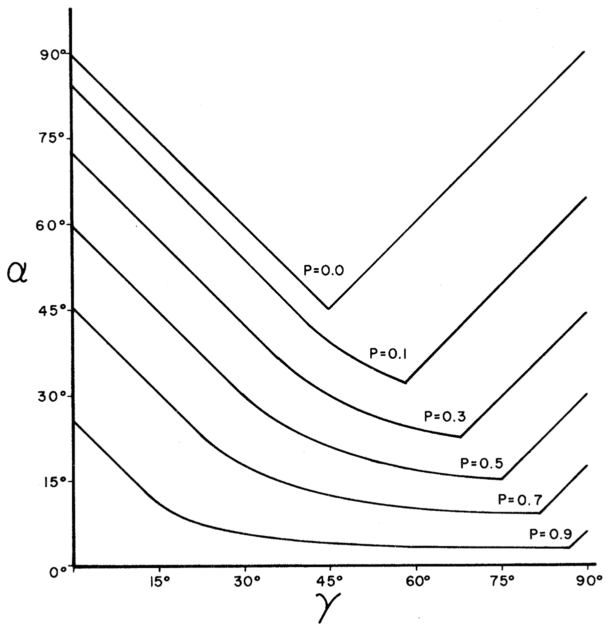

FIG. 3. Curves of constant P, the fraction of the sphere of velocity directions which cannot escape; γ is the angle the magnetic field makes with the minor axis of the toroid, and $\sin^2\alpha$ is the inverse of the mirror ratio.

where τ_{ii} is the ion-ion collision frequency and P is calculated from Eq. (15). In most cases of interest $\tau_c \gg \tau_g$ and therefore dominates. We are thus led to the conclusion that the time constant for the simple toroidally symmetric line cusp without an axial field is of the same order as that of a mirror device, although microinstabilities might alter this conclusion. The situation is, however, different in the Tormac[4,6] geometry: Because of the presence of the trapped B_ψ field, a time constant much longer than either for the mirror device or for the simple cusp results. In this geometry the outflow from the sheath is balanced by cross field diffusion from the plasma in the central region. On this basis a sheath thickness can be calculated, yielding a sheath narrower than the ion gyroradius. To be realistic, it is then safer to estimate the minimum sheath width as the ion gyroradius.

The time constant for the Tormac geometry is now given by multiplying the time constant for particle loss in the sheath by the ratio of the volume of the device to the volume of the sheath. Thus

$$\tau_{\text{Tormac}} \approx \tau_{ii} D/(1-P)2r_i, \tag{17}$$

where D is the minor diameter of the device and r_i the ion gyroradius.

In conclusion, the simple toroidally symmetric cusp with the assumptions discussed above[7] has a time constant of the same order of magnitude as a mirror device of the same size; the Tormac device has a time constant which is size dependent and longer than a mirror device. In a Tormac built for controlled thermonuclear fusion, D/r_i could be as large as 100, which implies a time constant with a similar factor larger than that of a mirror device.

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