obtains[4]

$$\Gamma(K^+ \to \pi^0\pi^0 e^+\nu) \approx (1.59\times10^3\ \mathrm{sec}^{-1})|f_1|^2, \qquad (2)$$

where f_1 is the axial-vector form factor multiplying the sum of the pion momenta in the matrix element. Another result of the $\Delta I=\frac{1}{2}$ rule is that f_1 for the decay $K^+\to\pi^0\pi^0e^+\nu$ is equal to the f_1 for the decay $K^+\to\pi^+\pi^-e^+\nu$.

From Eq. (2), the result of this experiment can be expressed as

$$|f_1| = 0.97^{+0.50}_{-0.19}.$$

The results of this experiment are in good agreement with the predictions of the $\Delta I=\frac{1}{2}$ rule. Our value for the magnitude of f_1 agrees with the value of 1.19 ± 0.13 obtained by Berends, Donnachie, and Oades,[5] from an analysis of $K^+\to\pi^+\pi^-e^+\nu$ data. The experimental $K^+\to\pi^+\pi^-e^+\nu$ branching ratio is reported[6] as $(3.3\pm0.3)\times10^{-5}$ and in a recent experiment[7] as $(4.11\pm0.38)\times10^{-5}$. Since a small rate for $K^0\to\pi^0\pi^-e^+\nu$ is predicted by several theoretical models for K_{e4} decays,[8] the experimental results are compatible with the $\Delta I=\frac{1}{2}$ prediction given by Eq. (1).

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Multiplicity Fluctuation and Multiparticle Distribution Functions in High-Energy Collisions*

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Speculations are made concerning the fluctuation in the multiplicity in the fragmentation of hadrons in high-energy collisions. We analyze implications for the two-particle distribution function ρ_2 and higher-order distributions ρ_ν.

The aim of this Letter is to stress the importance of the possible phenomena of wide multiplicity fluctuation in high-energy collisions and to speculate on some possible characteristic behavior of the two-particle distribution function. The subject is of immediate interest because a basic general feature of a number of current models in multiparticle production (statistical models, multiperipheral models) is the rough independence of the outgoing particles. This basic feature does not allow for a wide fluctuation of the multiplicity. On the other hand, the hypothesis of limiting fragmentation *does*, since fragmentation into very few fragments is envisaged as having a finite probability at very high energies, while the average multiplicity continues to increase with incoming energy.

We shall concentrate on discussing the charged-pion multiplicity n_{ch} since that is the simplest quantity to measure experimentally. Define

$$n_{\mathrm{ch}} = n_{\mathrm{ch}}{}^R + n_{\mathrm{ch}}{}^L, \qquad (1)$$

where R and L refer to the right and left hemispheres in the center-of-mass system for the outgoing momenta. We shall use the variable x defined by

$$x = p_\parallel{}^*/(p_\parallel{}^*)_{\mathrm{incoming}}, \qquad (2)$$

where $p_\parallel{}^*$ denotes the longitudinal momentum in the c.m. system. The hemisphere R is defined to be the one for which x is positive. For simplicity we shall integrate over all transverse momenta and concentrate on $d\sigma/dx_1$, $d^2\sigma/dx_1dx_2$, etc. and their limits at very high incoming energies:

$$(d\sigma/dx_1)\,dx_1 \to \rho_1(x_1)\,dx_1, \qquad (3)$$

$$(d^2\sigma/dx_1dx_2)dx_1dx_2 \to \rho_2(x_1,x_2)\,dx_1dx_2. \qquad (4)$$

Let x_1, x_2 refer to charged pions. (If the experiment is more specific, one can discuss the case when x refers to π^-'s, or to protons, etc.) It is clear that

$$\int_0^1 (d\sigma/dx_1)dx_1 = \sigma_T\langle n_{\rm ch}{}^R\rangle, \quad (5)$$

$$\int_{-1}^0 (d\sigma/dx_1)\,dx_1 = \sigma_T\langle n_{\rm ch}{}^L\rangle,$$

$$\int_0^1\int_0^1 (d\sigma/dx_1dx_2)\,dx_1dx_2 = \sigma_T\langle n_{\rm ch}{}^R(n_{\rm ch}{}^R-1)\rangle, \quad (6)$$

$$\int_0^1\int_{-1}^0 (d\sigma/dx_1dx_2)\,dx_1dx_2 = \sigma_T\langle n_{\rm ch}{}^R n_{\rm ch}{}^L\rangle, \quad (7)$$

etc. These are relations that follow from definitions and are true, independent of any specific assumptions.

Such quantities like $\langle(n_{\rm ch}{}^R)^2\rangle$ and $\langle n^2\rangle$ are clearly related to multiplicity fluctuations, e.g., $\langle(n-\langle n\rangle)^2\rangle = \langle n^2\rangle - \langle n\rangle^2$. Through Eqs. (6), (7), and their generalizations, one can also relate fluctuations to the two-particle distribution functions $d\sigma/dx_1dx_2$.

According to the hypothesis of limiting fragmentation, for a fixed $n_{\rm ch}{}^R$, the cross section $\sigma(n_{\rm ch}{}^R)$ approaches a limit at high energies. Assuming that $\langle n_{\rm ch}{}^R\rangle$ increases linearly with $\ln E_{\rm incoming}$, it is natural to assume

$$\lim_{E_{\rm inc}\to\infty} \sigma(n_{\rm ch}{}^R) = {\rm const}\,(n_{\rm ch}{}^R)^{-2} \quad \text{for large } n_{\rm ch}{}^R. \quad (8)$$

Speculations in this spirit lead us to the following guesses about multiplicity fluctuations and about the distribution function $d\sigma/dx_1dx_2$:

(a) The multiplicity fluctuation for the fragmentation of either hadron in the collision is large. If one tries for a *power fit* at high energies,

$$\langle(n_{\rm ch}{}^R)^2\rangle \propto (E_{\rm inc})^\alpha,$$

one easily proves that $\alpha \leq \frac{1}{2}$ if $\langle n_{\rm ch}\rangle \propto \ln E_{\rm inc}$. Equation (8) suggests that

$$\alpha = \tfrac{1}{2}. \quad (9)$$

Similarly, one can discuss the higher moments of $n_{\rm ch}{}^R$:

$$\langle(n_{\rm ch}{}^R)^3\rangle \propto (E_{\rm inc})^\beta.$$

We speculate that $\beta = 1$. Notice that for models where the independence of emission of outgoing particles is a more or less essential element of the basic picture, $\alpha=\beta=0$ if the average multiplicity increases like $\ln E_{\rm inc}$. [We take $\ln E_{\rm inc}$ to be $(E_{\rm inc})^0$.]

(b) The multiplicities of the fragmentation of the two hadrons are not very much correlated. In particular,

$$\langle n_{\rm ch}{}^R n_{\rm ch}{}^L\rangle / \langle n_{\rm ch}{}^R\rangle\langle n_{\rm ch}{}^L\rangle \to 1 \text{ as } E\to\infty. \quad (10)$$

If it is verified that indeed these statements are correct, then at high energies $\langle(n_{\rm ch}{}^R)^2\rangle$ is much larger than $\langle n_{\rm ch}{}^R n_{\rm ch}{}^L\rangle$. Thus, the high multiplicity at small x should perhaps be interpreted not as due to emission from some amalgamation of the two hadrons, but as due to the *separate* fragmentation of the two hadrons, excited in passing through each other.

If (10) is correct, it implies that the multiplicity fluctuation is *large within each hemisphere.* Measuring $\langle(n_{\rm ch})^2\rangle$, where $n_{\rm ch}$ is defined in (1), would *water down* the large fluctuation within each side, a point of some relevance if one wants to study this question experimentally.

(c) Let us define

$$\int_0^1 (d\sigma/dx_1dx_2)\,dx_2 = \rho_1(x_1)m^R(x_1). \quad (11)$$

$m^R(x_1)$ is then the average number[1] of additional charged particles in the right hemisphere (i.e., $x_2>0$) for all events where a charged particle with x_1 is known to be emitted. We speculate that for

$$x_1>0,\quad m^R(x_1)\to \text{a finite limit as } E\to\infty, \quad (12)$$

$$x_1<0,\quad m^R(x_1)\to\infty \text{ as } E\to\infty. \quad (13)$$

Statement (13) is easy to believe since the emission of a particle in the left hemisphere ($x_1<0$) is not expected to greatly affect the average multiplicity in the right hemisphere. In other words, one expects that, in fact, for

$$x_1<0,\quad m^R(x_1)\cong\langle n_{\rm ch}{}^R\rangle, \quad (13a)$$

which would imply (13). On the other hand, (12) represents a deviation[2] from the "independent-emission" type of concepts. What are then the reasons for the guess (12)? To answer this question let us remind ourselves first that $\langle n_{\rm ch}{}^R\rangle$ becomes increasingly larger at higher energies because $\sigma(n_{\rm ch}{}^R)$ does not vanish sufficiently fast for large $n_{\rm ch}{}^R$ [cf., e.g., (8) for which the relevant point is that

$$\sum^\infty \frac{n}{n^2}$$

is divergent]. But, if one insists on the condition of one charged particle emitted in the right hemisphere at $x_1>0$, the relevant cross section $\sigma(n_{\rm ch}{}^R)$ will have to be reduced by a factor $f(n_{\rm ch}{}^R, x_1)$ which depends on $n_{\rm ch}{}^R$. For very large $n_{\rm ch}{}^R$, the condition $\sum x=1$ over all positive x will make the emission of a single finite x_1 very unlikely. Thus, for large n, $f(n, x_1)$ decreases with n. A phase-space argument easily leads to an n dependence of the form

$$f(n, x_1) = n(n-1)(1-x_1)^{n-2}. \tag{14}$$

[See the mathematical model discussed below in (e).] Thus, $\sum \sigma(n) f(n, x)(n-1)$ would be convergent and we obtain (12).

Is (14) correct? It certainly is not exact. But, we believe a strong damping factor like $(1-x_1)^n$ is present for two reasons: (I) Under what circumstances would $f(n, x_1)$ not provide a strong damping factor as n becomes large? Only when for fixed and large n, the partition $\sum x = 1$ is such that one (or a few) pions would essentially always have a large x, while the rest would have $x = O(1/n)$. There seems to be very little theoretical reason for such a peculiar partition. (II) Experimentally, an examination of bubble-chamber pictures for high-energy collisions shows that for a multipion event there is no *persistent* trend to have such a peculiar partition. [That part of the single-pion distribution $\rho_1(x)$ which is at large x comes from events with limited multiplicity n.]

If we go to the limiting distributions (3) and (4), we would have, according to (12) and (13),

$$\int_0^1 \rho_2(x_1, x_2)\, dx_2 = \infty, \quad x_1 < 0$$
$$= \text{a finite function of } x_1, \quad x_1 > 0. \tag{15}$$

(d) This last equation suggests that

$$\rho_2(x_1, 0+) = \text{a finite function of } x_1, \quad x_1 > 0. \tag{16}$$

Indeed, the discussions under (c) that led to (12) strongly suggest (16). The point is that the divergence of $\rho_1(x_1)$ as $x_1 \to 0+$ is due to high-multiplicity events. In $\rho_2(x_1, 0+)$ the specification of one pion at x_1 damps the probability of the high-multiplicity event so that $\rho_2(x_1, 0+)$ is finite. How could (15) be in accord with (6) and (9)? Clearly, the answer lies in the strong divergence of $\rho_2(0+, 0+)$. For example, one possibility is that it has a singularity

$$\rho_2(x_1, x_2) \approx \text{const}\,(x_1 + x_2)^{-3}$$
$$\text{for } x_1 \cong 0+,\ x_2 \cong 0+. \tag{17}$$

Further,

$$m^R(x_1) \approx \text{const}\, x_1^{-1} \text{ for } x_1 \cong 0+. \tag{18}$$

Notice that (13a) suggests that for $x_1 < 0$, $\rho_2(x_1, 0+)$ has a singularity $\sim \text{const}\, x_2^{-1}$. Thus $\rho_2(x_1, x_2)$ *has discontinuities at* $x_1 = 0$ and at $x_2 = 0$.

(e) We are of the opinion that at this time any specific detailed theoretical models are of little use except to illustrate some general qualitative features. In this spirit one can construct a very simple model[3] in which there is only one kind of particle and, writing l for n^R, we assume, for infinite energy,

$$\sigma(l) = K/l(l-1), \quad l \geq 2, \quad \sigma(1) = 0, \tag{19}$$

which satisfies (8). We then assume that for given l the longitudinal momentum distribution is

$$\left(\prod^l dx_i\right)\delta\left(1 - \sum x_i\right),$$

(i.e., strictly a phase space in x variables). The computation of ρ_1, ρ_2, etc. is straightforward,[4] yielding for this model,

$$\rho_1(x) = Kx^{-1}, \quad \sigma_T = K, \tag{20}$$

$$\rho_2(x, y) = 2K(x+y)^{-3} + K\delta(1-x-y),$$
$$\text{for } x > 0,\ y > 0, \tag{21}$$

$$m^R(x) = x^{-1}. \tag{22}$$

(f) All of the above points can be generalized to $\rho_3, \rho_4 \cdots$. Let us just mention that we speculate that $\rho_\nu(x_1, x_2, \cdots, x_\nu)$ is finite inside and on the boundary of the region $x_1 > 0$, $x_2 > 0, \cdots, x_\nu > 0$ except at the origin. Near the origin it has a strong singularity of the form

$$\rho_\nu \propto \text{const}\left(\sum^\nu x\right)^{-(2\nu - 1)}, \quad x_i > 0. \tag{23}$$

Does existing experimental information confirm or contradict the above speculations? We know of no data which lend conclusive support to these speculations, but because the mean multiplicities observed in present-day experiments are so low, definitive statements are not expected to emerge. We do wish to cite a few experimental statements which, taken together, indicate to us that our conjectures stand a chance of being correct.

(i) That the large-x portion of the single-pion distribution $\rho_1(x)$ is contributed by events of limited multiplicity n is indicated by many experiments. A particularly clear demonstration occurs in Fig. 2 of Biswas *et al.*[5]

(ii) According to the reasoning outlined in (d) and (e) above, $\rho_2(x, y)$ should depend on x and y mainly in the combination $x+y$, when x and y are in the same hemisphere. A strong qualitative feature of experimental two-particle distribution functions in a variety of reactions at several energies is indeed the constancy of $\rho_2(x, y)$ along lines of fixed $x+y$ when two particles are observed in the same hemisphere.[6-8]

(iii) The discussion leading to (14) and the results (20) and (21) imply that for fixed and small values of x_2 (in the same hemisphere as x_1), $\rho_2(x_1, x_2)$ is more sharply peaked near $x_1 = 0$ than is $\rho_1(x)$, in agreement with the data cited in (ii) above.

(iv) The rather low multiplicity data available to us[7] are not inconsistent with the behavior suggested by (13a) and (18) for the (right) hemisphere associated multiplicity $m^R(x)$.

(v) Our results (20) and (21) imply that at high energies the so-called correlation function

$$C(x, y) \equiv \rho_2(x, y) - \rho_1(x)\rho_1(y)\sigma_T^{-1} \tag{24}$$

should behave as

$$\lim_{E_{\text{inc}} \to \infty} C(0+, 0+) \to +\infty. \tag{25}$$

The existing data[6-9] indeed suggest the development of such a positive spike in $C(x, y)$ with increasing energy. In this connection we emphasize that for many purposes it is useful to define the correlation function in another way, namely, as $Z(x, y) = \rho_2(x, y)/(\sigma_T^{-1})\rho_1(x)\rho_1(y)$. The divergence discussed above is much stronger than that in a model such as the multiperipheral model.[10] The singularity of $\rho_2(x, y)$ near $x = y = 0$ is closely related to $\langle n^2 \rangle$, as discussed above. We believe that *this singularity exists and is quite strong*, even if $\langle n^2 \rangle$ should turn out to be not as large as indicated by the guess (9). Also, we believe the *stronger dependence on the variable* $x + y$ *than* $x - y$ (when both x and y are in the same hemisphere), especially near the singularity at the origin, is likely to be correct.

(vi) The existence of events with very large multiplicities in cosmic rays together with the slow rise of the average multiplicity with increasing energy suggests that the fluctuation in multiplicity is large.

Our interest in the phenomena discussed here was stimulated by our colleagues in the Pisa-Stony Brook collaboration working at the intersecting storage rings. We are grateful to them for showing us their preliminary data and for making us aware of the capacity of their experimental setup to resolve the kinds of questions we pose. We are also happy to thank Professor Margaret Foster, Professor William Shephard, and Dr. Sheldon Stone for sharing and discussing their experimental results with us.

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[1]A similar quantity $m^L(x_1)$ could be defined with the integral extending between -1 and 0. Notice that $m(x_1) = m^R(x_1) + m^L(x_1)$ is the associated multiplicity discussed in H. T. Nieh and J. M. Wang, Phys. Rev. D (to be published).

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$$\rho_2 = \sum_l^\infty \rho_{2l} + \rho_{22},$$

where ρ_{22} is a δ function.

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