E_z and H_z, and that in the second case there are no interference terms between the orthogonal modes L_y and R_y. We thus conclude that *the image of the tangential focal line is a doublet, one line having an* $E_\perp$ *and the other an* $E_\parallel$ *polarization, and that the image of the sagittal focal line is also a doublet, the two lines having opposite circular polarizations inside the evanescent wave.*

One of us and co-workers[8] have proved experimentally the first property, and we also hope to test the second one soon.

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Uniqueness of the Vacuum Energy Density and van Hove Phenomenon in the Infinite-Volume Limit for Two-Dimensional Self-Coupled Bose Fields*

Francesco Guerra†
Joseph Henry Laboratories of Physics, Princeton University, Princeton, New Jersey 08540
(Received 9 February 1972)

For a class of model quantum field theories in two-dimensional space-time describing a neutral scalar boson field interacting in a region of length l, the vacuum energy density converges as $l\to\infty$. In the same limit the vacuum state approaches zero in the weak sense, as proposed by van Hove.

We consider a self-coupled scalar field in two-dimensional space-time with total Hamiltonian[1-4]

$$H_l = H_0 + V_l,\qquad V_l = \int_{-l/2}^{l/2} :P(\varphi(x)):\,dx, \tag{1}$$

where $P(\lambda)$ is a polynomial with real coefficients, bounded below and normalized to $P(0)=0$.

Our results are the following: (a) The vacuum energy E_l of H_l has an upper bound linear in the size l of the interaction region. (b) The vacuum energy density $\alpha(l)=|E_l|/l$ has a unique finite limit when $l\to\infty$. (c) The normalized approximate vacuum Ω_l goes weakly to zero in the Φοκ space when $l\to\infty$ (van Hove phenomenon[5]). These results hold in perturbation theory but they need a proof independent of the perturbative expansion because this is known to diverge in this case.[6]

According to one of the methods previously reported,[1,3,4,7] we consider the representation of the Φοκ space as $L^2(Q,\mu)$, where (Q,μ) is a probability space, the bare vacuum (no-particle state) Ω_0 is then represented by the function 1 on Q and $V_l \in L^p(Q,\mu)$, $p<\infty$. The Hamiltonian H_l can be defined as an operator essentially self-adjoint on $D(H_0)\cap D(V_l)$, with a unique eigenstate Ω_l of lowest energy E_l. With a convenient choice of normalization and phase factor, one has $\|\Omega_l\|_2=1$, $\Omega_l>0$ almost everywhere on Q; and moreover $\Omega_l \in L^p(Q,\mu)$ for any $p<\infty$.[4] If $l>0$, then $\Omega_l\neq\Omega_0$, $E_l<0$, and $\|\Omega_l\|_1<1$.

Lemma I.—Let H_l and Ω_0 be defined as above; then the equality

$$\langle\Omega_0, \exp(-tH_l)\Omega_0\rangle = \langle\Omega_0, \exp(-lH_t)\Omega_0\rangle \tag{2}$$

holds for any $t, l\geq 0$. This result is due to Nelson[8]; we sketch a proof at the end of this note. An immediate consequence of lemma I is the following.

Theorem I.—There are positive constants α and β such that

$$E_l \leq -\alpha l + \beta. \tag{3}$$

Moreover, the vacuum energy density $\alpha(l)=|E_l|/l$ has a unique limit when $l\to\infty$.

Remark.—The analogous linear lower bound has been established by Glimm and Jaffe[9] and plays a very important role in the control of the infinite-volume limit of the theory.

Proof of theorem I.—Let P_t be the projection operator on Ω_t; then one has

$$\begin{aligned}\exp(-lE_t)\langle\Omega_0, P_t\Omega_0\rangle &= \langle\Omega_0, \exp(-lH_t)P_t\Omega_0\rangle\\ &\leq \langle\Omega_0, \exp(-lH_t)\Omega_0\rangle\\ &= \langle\Omega_0, \exp(-tH_l)\Omega_0\rangle\\ &\leq \exp(-tE_l).\end{aligned}$$

Therefore, since $\langle\Omega_0, P_t\Omega_0\rangle = \|\Omega_t\|_1{}^2$, then

$$-lE_t + \ln\|\Omega_t\|_1{}^2 \leq -tE_l \tag{4}$$

and the first part of the theorem is established.

To prove the second part, let us define, for $l > 0$,

$$\alpha(l) = -E_l/l = |E_l|/l, \qquad \beta(l) = -l^{-1}\ln\|\Omega_l\|_1{}^2. \tag{5}$$

Then the previous inequality gives us

$$\alpha(l) \geq \alpha(t) - l^{-1}\beta(t). \tag{6}$$

Now define

$$\alpha_\infty = \sup_{l \geq l_0}\alpha(l),$$

for some $l_0 > 0$. Because of the linear lower bound on E_l, α_∞ is finite. It is evident from (6) that

$$\lim_{l\to\infty}\alpha(l) = \alpha_\infty,$$

and the theorem is proved.

Our second main result is the following theorem.

Theorem II.—The van Hove phenomenon holds in L^2, i.e., $\Omega_l \to 0$ weakly in $L^2(Q, \mu)$ as $l\to\infty$.

Proof of theorem II.—Consider the eigenvalue equation for $l > 0$,

$$(H_0 + V_l)\Omega_l = E_l\Omega_l, \tag{7}$$

and take the scalar product with Ω_0. Since $\Omega_0 \in D(H_0)\cap D(V_l)$, $H_0\Omega_0 = 0$, and $\Omega_0 = 1$ on Q, one gets

$$\int_Q V_l\Omega_l\, d\mu = E_l\|\Omega_l\|_1. \tag{8}$$

Therefore $\|\Omega_l\|_1 \leq |E_l|^{-1}\|V_l\|_2$.

Now theorem I tells us that, for l large enough, there is a constant $\alpha' > 0$ such that $|E_l| > \alpha' l$. On the other hand, there is a constant $c > 0$ such that

$$\|V_l\|_2{}^2 \leq cl. \tag{9}$$

Therefore one has $\|\Omega_l\|_1 \to 0$ as $l\to\infty$. To complete the proof it is necessary to show that $\langle\psi, \Omega_l\rangle \to 0$ as $l\to\infty$, for any $\psi\in L^2(Q,\mu)$. Since L^∞ is dense in L^2 in the L^2 norm, there is a $u\in L^\infty$ such that $\|\psi - u\|_2 < \frac{1}{2}\epsilon$ for a given $\epsilon > 0$. But $\lim_{l\to\infty}\langle u, \Omega_l\rangle = 0$ because $u \in L^\infty$ and $\|\Omega_l\|_1 \to 0$; therefore there is an l_0 such that $|\langle u, \Omega_l\rangle|_1 < \frac{1}{2}\epsilon$ for any $l > l_0$. Therefore for $l > l_0$ one has $|\langle\psi, \Omega_l\rangle| \leq |\langle\psi - u, \Omega_l\rangle| + |\langle u, \Omega_l\rangle| < \epsilon$, and the theorem is proved.

Remarks: (A) The falloff of $\|\Omega_l\|_1$ as $l\to\infty$ can be easily generalized. For any p, $1 \leq p < 2$, and any $k > 0$, there is a constant $c(p;k)$ such that

$$\|\Omega_l\|_p \leq c(p;k)l^{-k}. \tag{10}$$

(B) A stronger statement follows from the assumption that E_l/l is not a constant. In this case, using (4), it is very easy to prove that for any p, $1 \leq p < 2$, there are constants $\alpha_p > 0$ and $\beta_p > 0$ such that

$$\|\Omega_l\|_p \leq \alpha_p\exp(-\beta_p l). \tag{11}$$

(C) Using the Riesz interpolation theorem it is very easy to prove that $\|\Omega_l\|_p \to \infty$ when $l\to\infty$, if $p > 2$.

Now we sketch a proof of lemma I. The simplest way is to exploit the connection between the Euclidean theory of Markov fields and the Hamilton formalism recently discovered by Nelson.[8] Let $\hat\psi(f)$ $[f\in \mathcal{S}(R^2)]$ be the Euclidean-Markov-Gaussian field defined by the expectation values $E(\hat\psi(f)) = 0$ and $E(\hat\psi(f)\hat\psi(g)) = \hat S(f,g)$, where $\hat S$ is the Euclidean propagator in two-dimensional space-"time."[10] Then it is not difficult to prove[8] that

$$\langle\Omega_0, \exp(-tH_l)\Omega_0\rangle = E\{\exp[\int_0^{\pm}\int_{-l/2}^{l/2} :P(\hat\psi(\xi)):\, d^2\xi]\}. \tag{12}$$

This formula can be considered as a version of the well-known Feynman-Kac formula (see Ref. 9, and earlier references quoted there) written in terms of the fields $\hat\psi$. Lemma I follows immediately from the invariance of the expectation value under the Euclidean group $E(2)$.

An alternative proof might possibly be obtained using the Lorentz invariance of the free theory and the locality of the interaction.

Recently Osterwalder and Schrader[11] have obtained some general results which, in the particular case at hand, go in the direction of the proof of the second part of theorem I. They have isolated two properties [P and S, see (11) for the definition] of the vacuum energy which imply the uniqueness of the infinite-volume limit of the energy density. Property P is known to hold from the work of Glimm and Jaffe[12]; property S has been proved[11] in a simplified model where the free energy H_0 is replaced by the number operator N.

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†On leave from Istituto di Fisica Teorica, Università di Napoli and Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Italy.

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Missing-Mass Spectra Produced by 2-GeV Protons in the Reaction $p + d \to He^3 + X^0$

H. Brody,* E. Groves,*† R. Van Berg,* and W. Wales*
University of Pennsylvania, Philadelphia, Pennsylvania 19104

and

B. Maglich,‡ J. Norem,‡§ and J. Oostens‡‖
Rutgers, The State University, New Brunswick, New Jersey 08903

and

G. B. Cvijanovich
Upsala College, East Orange, New Jersey 07019
(Received 31 January 1972)

We investigated the missing-mass spectra at the 2π threshold in the reaction $p+d\to He^3+X^0$ and searched for the resonance reported at a mass of 450 MeV using 2-GeV protons. The shape of the missing-mass spectra below the η^0 mass deviates from two-body phase space and was similar to that obtained with 3-GeV protons. If a 450-MeV resonance exists, the c.m. cross section for producing it in this reaction is less than 10.8×10^{-36} and 5.7×10^{-36} cm^2/sr with 2- and 3-GeV protons, respectively.

The reaction $p+d\to He^3+X^0$ has consistently shown an enhancement in the missing-mass spectrum above the two-body relativistic invariant phase space at the two-pion threshold.[1,2] The similar reaction $d+p\to He^3+X^0$ also shows an enhancement at a missing mass of 300 MeV and, in addition, a bump at a mass of 450 MeV.[3] To investigate both of these phenomena, the energy of the Princeton-Pennsylvania accelerator was reduced from 3 to 2 GeV to increase the counting rate and improve the mass resolution. This reduction in energy increased the He^3 counting rate by a factor of ≈ 10 and improved the missing-mass resolution (squared) from $\Delta M^2 = 0.023 \pm 0.003$ $(\mathrm{GeV}/c^2)^2$ to $\Delta M^2 = 0.017 \pm 0.002$ $(\mathrm{GeV}/c^2)^2$. The equipment and technique used are the same as previously reported[2,4] with the Jacobian-Peak method and detection of only the recoil He^3.

The 2-GeV data [Fig. 1(a)] clearly show the increase in cross section and improvement in resolution over the data taken at 3 GeV [Fig. 1(b)]. There are approximately 1500 events per 0.0075-$(\mathrm{GeV}/c^2)^2$ mass bin in the 300- to 500-MeV mass range. The only allowed strong reactions that