TABLE I. Forward nuclear scattering-amplitude parameters from Eq. (2).

E (MeV)	$\alpha(0)$	R_S [a] (fm)	K (fm^2)	χ^2/n [b]	σ_T [c] (mb)	Im$f_N(0)$ [d] (fm)	Re$f_N(0)$ (fm)
115 [e]	0.23 ± 0.06	3.49	27 ± 11	0.68	675 ± 8	5.83 ± 0.07	1.34 ± 0.35
115	0.24 ± 0.06	3.49	0	1.58	675 ± 8	5.83 ± 0.07	1.40 ± 0.35
167	$0.03^{+0.09}_{-0.06}$	3.32	0	0.88	685 ± 8	7.50 ± 0.09	$0.23^{+0.68}_{-0.45}$
242	-0.28 ± 0.04	3.18	0	1.30	565 ± 7	8.10 ± 0.10	-2.27 ± 0.33

[a]Values interpolated from π^--^{12}C amplitude fits from Binon *et al.* (Ref. 1).
[b]$n = 12$ is the number of experimental points fitted.
[c]Values interpolated from π^--^{12}C total cross sections from Binon *et al.* (Ref. 6).
[d]Im$f_N(0) = P\sigma_T/4\pi$.
[e]K allowed to vary.

the latter. Since Re$f_N(0)$ is consistent with zero at 167 MeV, the $(\frac{3}{2}, \frac{3}{2})$ resonance energy for pion-carbon scattering must be near this energy. This result is in disagreement with a Glauber-model treatment, which predicts Re$f_N(0) = 0$ at the π-nucleon resonance energy.

Calculations of the Kisslinger optical model[2] using interpolated parameters from fits to π^--^{12}C large-angle data[6] are indicated in Fig. 2(b) by the solid curves. The optical model gives qualitative agreement with these small-angle cross sections except near the $(\frac{3}{2}, \frac{3}{2})\pi^{\pm}$-^{12}C resonance energy. Near the resonance energy, it is difficult to interpolate the rapidly varying $l = 1$ parameters in the model, and this is the probable cause of the disagreement.

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Quantized Longitudinal and Transverse Shifts Associated with Total Internal Reflection

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A recent experiment has shown us that the Goos-Hänchen shift is quantized, the eigenfunctions being the transverse electric and magnetic modes. We give here a straightforward theory of this phenomenon, together with the prediction that our new transverse shift is also quantized, with the circularly polarized modes inside the evanescent wave as eigenfunctions. An experimental test of the latter point is being considered.

First we make it clear that the now well-known Goos-Hänchen[1] longitudinal shift, and the new transverse shift one of us[2] has calculated and proved experimentally, are not simultaneously observable. The existence of the Goos-Hänchen shift entails that total reflection at a plane interface is *not* optically stigmatic, so that the image of a point source S as produced by a pencil of rays consists of a tangential focal line[3] orthogonal to the mean incidence plane, and a sagittal focal line[3] lying along the perpendicular to the reflecting plane containing S. Obviously, observing the Goos-Hänchen shift is equivalent to observing the tangential focal line, and observing our[2] transverse shift is equivalent to observing the sagittal focal line. As these lines cannot be brought simultaneously into focus, *the longitudinal and the transverse shifts associated with total reflection are not simultaneously observable.*

Using an energy-flux conservation argument, Renard[4] wrote the formula of the longitudinal Goos-Hänchen shift X as

$$c^{-1}\omega X = C_1(\tau_\perp{}^*\tau_\perp E_\perp{}^* E_\perp + \tau_\parallel{}^*\tau_\parallel E_\parallel{}^* E_\parallel), \qquad (1)$$

where ω denotes the angular frequency, $E_\perp$ and $E_\parallel$ are the electric field strengths perpendicular and parallel to the incidence plane inside the incident plane wave, $\tau_\perp$ and $\tau_\parallel$ are Fresnel's transmission coefficients, and C_1 is a numerical constant depending on the index and the incidence angle. Similarly one of us[2] has written for the transverse shift Z a formula which reads, after substitutions,

$$\sqrt{2}\, L = e_\perp E_\perp + i e_\parallel E_\parallel, \quad \sqrt{2}\, R = e_\perp E_\perp - i e_\parallel E_\parallel, \qquad (2)$$

$$\sqrt{2}\, e_\perp E_\perp = L + R, \quad i\sqrt{2}\, e_\parallel E_\parallel = L - R, \qquad (3)$$

where $e_\perp$ and $e_\parallel$ denote the phase factors of $\tau_\perp$ and $\tau_\parallel$, C_2 denoting a numerical constant,

$$c^{-1}\omega Z = C_2 |\tau_\perp{}^*\tau_\parallel| (R^*R - L^*L). \qquad (4)$$

The normalization condition

$$E_\perp{}^*E_\perp + E_\parallel{}^*E_\parallel = L^*L + R^*R = 1 \qquad (5)$$

is assumed in (1) and (4); the L and R modes are orthogonal ($R^*L = 0$) if and only if $E_\parallel = \pm i E_\perp$; they are then the circularly polarized modes *inside the evanescent wave.*

The point is that *both formulas (1) and (4) have the canonical form of a quantum-mechanical mean value, with eigenvalues and eigenfunctions displayed.* Assuming that X and Z are quantized according to this scheme, it is a straightforward matter to write down the corresponding operators in diagonalized form, and also the unitary transformation between the two complementary descriptions

$$|\mathcal{L}\rangle = \begin{pmatrix} E_\perp \\ E_\parallel \end{pmatrix} \quad \text{and} \quad |\mathcal{C}\rangle = \begin{pmatrix} L \\ R \end{pmatrix}.$$

This we refrain from doing for lack of space.

Now we *prove* that X and Z are indeed quantized quantities. The assumption is that, when calculating the images of the tangential and the sagittal focal lines, the quantity of significance is Poynting's energy-flux vector; the justification for this is that formulas (1) and (4), derived through Poynting's theorem, are experimentally excellent, and surprisingly good theoretically (as we have just seen).

It happens that one of us has already given compact formulas for the evanescent wave *in vacuo* in the two cases, presently of significance, of a Fourier expansion on k_x and k_y ($k_z \equiv 0$),[5] and on k_x and k_z [$k_y = -\omega(n^2\alpha^2 - 1)^{1/2} y$; n, refracting index; α, sine of the incidence angle].[6] In both cases the time dependence is taken as $\exp(i\omega t)$. In the first case the field strengths depend on the transverse electric $E_z(x,y)$ and the transverse magnetic $H_z(x,y)$ modes that are solutions of the Helmholtz equation

$$(\partial_x{}^2 + \partial_y{}^2 + \omega^2)|\mathcal{L}\rangle = 0. \qquad (6)$$

In the second case they depend on the components L_y and R_y of the circularly polarized modes $\vec{L}$ and $\vec{R}$ inside the evanescent wave that are solutions of

$$(\partial_x{}^2 + \partial_z{}^2 + n^2\alpha^2\omega^2)|\mathcal{C}\rangle = 0. \qquad (7)$$

We have set[7]

$$\sqrt{2}\,\vec{L} = \vec{E} + i\vec{H}, \quad \sqrt{2}\,\vec{R} = \vec{E} - i\vec{H}, \qquad (8)$$

$$\sqrt{2}\,\vec{E} = \vec{L} + \vec{R}, \quad i\sqrt{2}\,\vec{H} = \vec{L} - \vec{R}. \qquad (9)$$

Then, calculating the Poynting vector

$$\vec{S} = \tfrac{1}{4}[\vec{E}^*\times\vec{H} + \vec{E}\times\vec{H}^*] = \tfrac{1}{4}i[\vec{R}^*\times\vec{R} - \vec{L}^*\times\vec{L}], \qquad (10)$$

we obtain in the first case

$$S_{x,y} = (i/4\omega)\{E_z{}^*[\partial_{x,y}]E_z + H_z{}^*[\partial_{x,y}]H_z\}, \qquad (11)$$

and in the second case

$$\begin{aligned} S_x &= (i/4n^2\alpha^2\omega)\{L_y{}^*[\partial_x]L_y + R_y{}^*[\partial_x]R_y - (n^2\alpha^2-1)^{1/2}(L_y{}^*[\partial_z]L_y - R_y{}^*[\partial_z]R_y)\}, \\ S_z &= (i/4n^2\alpha^2\omega)\{L_y{}^*[\partial_z]L_y + R_y{}^*[\partial_z]R_y + (n^2\alpha^2-1)^{1/2}(L_y{}^*[\partial_x]L_y - R_y{}^*[\partial_x]R_y)\}; \end{aligned} \qquad (12)$$

$i[\partial] \equiv i(\vec{\partial} - \overleftarrow{\partial})$ denotes the Schrödinger or Gordon current operator. In the second case the vector $(S_x, 0, S_z)$ is the sum of longitudinal and transverse Schrödinger-like currents, the latter explaining our[2] transverse shift.

The point is that in the first case there are no interference terms between the orthogonal modes

E_z and H_z, and that in the second case there are no interference terms between the orthogonal modes L_y and R_y. We thus conclude that *the image of the tangential focal line is a doublet, one line having an* $E_\perp$ *and the other an* $E_\parallel$ *polarization, and that the image of the sagittal focal line is also a doublet, the two lines having opposite circular polarizations inside the evanescent wave.*

One of us and co-workers[8] have proved experimentally the first property, and we also hope to test the second one soon.

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Uniqueness of the Vacuum Energy Density and van Hove Phenomenon in the Infinite-Volume Limit for Two-Dimensional Self-Coupled Bose Fields*

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For a class of model quantum field theories in two-dimensional space-time describing a neutral scalar boson field interacting in a region of length l, the vacuum energy density converges as $l\to\infty$. In the same limit the vacuum state approaches zero in the weak sense, as proposed by van Hove.

We consider a self-coupled scalar field in two-dimensional space-time with total Hamiltonian[1-4]

$$H_l = H_0 + V_l,\qquad V_l = \int_{-l/2}^{l/2} :P(\varphi(x)):\,dx, \tag{1}$$

where $P(\lambda)$ is a polynomial with real coefficients, bounded below and normalized to $P(0)=0$.

Our results are the following: (a) The vacuum energy E_l of H_l has an upper bound linear in the size l of the interaction region. (b) The vacuum energy density $\alpha(l) = |E_l|/l$ has a unique finite limit when $l\to\infty$. (c) The normalized approximate vacuum Ω_l goes weakly to zero in the $\Phi\circ\kappa$ space when $l\to\infty$ (van Hove phenomenon[5]). These results hold in perturbation theory but they need a proof independent of the perturbative expansion because this is known to diverge in this case.[6]

According to one of the methods previously reported,[1,3,4,7] we consider the representation of the $\Phi\circ\kappa$ space as $L^2(Q,\mu)$, where (Q,μ) is a probability space, the bare vacuum (no-particle state) Ω_0 is then represented by the function 1 on Q and $V_l \in L^p(Q,\mu)$, $p<\infty$. The Hamiltonian H_l can be defined as an operator essentially self-adjoint on $D(H_0)\cap D(V_l)$, with a unique eigenstate Ω_l of lowest energy E_l. With a convenient choice of normalization and phase factor, one has $\|\Omega_l\|_2 = 1$, $\Omega_l > 0$ almost everywhere on Q; and moreover $\Omega_l \in L^p(Q,\mu)$ for any $p<\infty$.[4] If $l>0$, then $\Omega_l \ne \Omega_0$, $E_l<0$, and $\|\Omega_l\|_1 < 1$.

Lemma I.—Let H_l and Ω_0 be defined as above; then the equality

$$\langle\Omega_0, \exp(-tH_l)\Omega_0\rangle = \langle\Omega_0, \exp(-lH_t)\Omega_0\rangle \tag{2}$$

holds for any $t, l \geq 0$. This result is due to Nelson[8]; we sketch a proof at the end of this note. An immediate consequence of lemma I is the following.

Theorem I.—There are positive constants α and β such that

$$E_l \leq -\alpha l + \beta. \tag{3}$$

Moreover, the vacuum energy density $\alpha(l) = |E_l|/l$ has a unique limit when $l\to\infty$.

Remark.—The analogous linear lower bound has been established by Glimm and Jaffe[9] and plays a very important role in the control of the infinite-volume limit of the theory.

Proof of theorem I.—Let P_t be the projection