formly decelerated. If the container decelerates smoothly with occasional glitches, it would be suggestive that the metastable fluid mechanism explained the pulsar behavior. We are currently studying the feasibility of such experiments. It is certainly appealing to hope one can make a laboratory analog of a neutron star and hence study some aspects of a system so completely inaccessible to direct experiment.

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Role of Conformal Three-Geometry in the Dynamics of Gravitation*

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The unconstrained dynamical degrees of freedom of the gravitational field are identified with the conformally invariant three-geometries of spacelike hypersurfaces. New results concerning the action principle, choice of canonical variables, and initial-value equations strengthen this identification. One of the new canonical variables is shown to play the role of "time" in the formalism.

An increasing amount of evidence shows that the true dynamical degrees of freedom of the gravitational field can be identified directly with the conformally invariant geometry of three-dimensional spacelike hypersurfaces embedded in space-time. It is the purpose of this paper to describe some of the new results that contribute to this evidence. The picture of dynamics that emerges is of the time-dependent geometry of shape ("transverse modes") interacting with the changing scale of space ("longitudinal mode"). At a moment when the three-geometry is maximal ($\pi = 0$), this interaction turns off and the dynamics becomes particularly amenable to analysis, as described below. We begin by briefly recalling the conclusions which led to singling out conformal three-geometry. Then new results are described concerning the initial-value equations, choice of canonical variables, a canonical "time" coordinate, and properties of the action integral of general relativity. We conclude by tying in a number of recent findings of other workers.

It has long been known that general solutions of the initial-value equations can be obtained when the metric γ_{ab} of the initial spacelike hypersurface is specified only up to an initially unknown conformal factor.[1-3] This means that only the conformal metric $\tilde{\gamma}_{ab} \equiv \gamma^{-1/3}\gamma_{ab}$ is freely specified, inasmuch as it is invariant with respect to conformal transformations $\gamma_{ab} \to \varphi^4\gamma_{ab} = \tilde{\gamma}_{ab}$, with $\varphi(x)$ arbitrary. The initially unknown conformal factor $\varphi(x)$ is found as part of the complete solution of the initial-value equations. The coordinate-independent concept behind the conformal metric is the conformal geometry $\tilde{\mathcal{G}}_3$, defined as the conformal equivalence class of diffeomorphically equivalent Riemannian three-metrics. The geometry $\tilde{\mathcal{G}}_3$ concerns only local angles and directions, but not distances; it is dimensionless.

The question of the physical significance of the conformal technique was recently answered.[4,5] It was established that the three-dimensional conformal curvature tensor β^{ab} is symmetric, traceless, and covariantly transverse ($\nabla_b\beta^{ab}=0$). The $\frac{5}{3}$-weight form of this tensor is given by

$$\tilde{\beta}^{ab}=\tfrac{1}{2}\gamma^{1/3}(\epsilon^{efa}\gamma^{bm}+\epsilon^{efb}\gamma^{am})\nabla_e R_{fm}, \tag{1}$$

where ϵ^{efa} is the unit alternating tensor. The above properties of $\tilde{\beta}^{ab}$, together with the fact that it is conformally invariant, show that conformally equivalent three-geometries give equivalent pure spin 2 (i.e., transverse and traceless) representations of the gravitational field. This led to the identification of $\tilde{\mathcal{G}}_3$ with the true unconstrained degrees of freedom.

In terms of the standard ADM canonical variables,[6] the initial-value equations have the form

$$\mathcal{H}^a\equiv-2\nabla_b\pi^{ab}=0, \tag{2}$$

$$\mathcal{H}\equiv\gamma^{-1/2}(\pi_{ab}\pi^{ab}-\tfrac{1}{2}\pi^2)-\gamma^{1/2}R=0. \tag{3}$$

Consider these equations on a maximal hypersurface ($\pi=0$). Then Eqs. (2) are invariant with respect to the substitutions $\tilde{C}$: $\gamma_{ab}\to\varphi^4\gamma_{ab}$, $\pi^{ab}\to\varphi^{-4}\pi^{ab}$, where $\varphi(x)$ is arbitrary. However, Eq. (3) is not invariant. It assumes the form[4]

$$\nabla^2\varphi+\tfrac{1}{8}M\varphi^{-7}-\tfrac{1}{8}R\varphi=0, \tag{4}$$

which *determines* φ. The quantity $M\equiv\gamma^{-1}\pi_{ab}\pi^{ab}$ is a non-negative function and π^{ab} is assumed to satisfy $\mathcal{H}^a=0$. In an important recent paper,[7] it was shown by Choquet-Bruhat that solutions to (4) exist and are unique on both closed and open manifolds, appropriate boundary conditions being assumed in the latter case. Here let us consider closed three-manifolds, i.e., compact manifolds without boundary. Then we can give a very simple proof of uniqueness[8] as follows: Suppose that (2) and (3) are satisfied on a maximal hypersurface. If we make an infinitesimal $\tilde{C}$ transformation, $\delta\gamma_{ab}=\lambda\gamma_{ab}$, $\delta\pi^{ab}=-\lambda\pi^{ab}$, then we find $\delta\mathcal{H}^a=0$ and $\delta\pi=0$ for an *arbitrary* function λ ($\lambda^2\approx0$). But $\delta\mathcal{H}=\gamma^{1/2}(2\nabla^2\lambda-2\lambda R)$ if $\mathcal{H}=0$. There is *no* λ which can make $\delta\mathcal{H}=0$, for it would have to satisfy

$$\nabla^2\lambda=R\lambda. \tag{5}$$

From (3) and $\pi=0$ we know that $R>0$, which means that there can be no solutions to (5) in the present case, as is readily seen by multiplying each side of (5) by λ and integrating over the three-space. The restriction to infinitesimal $\tilde{C}$ transformations is only for convenience; the finite case gives the same result. Moreover, these conclusions can be readily generalized in two ways. Firstly, matter sources $\mathcal{T}_\nu{}^\mu(x)$ can be inserted on the right-hand sides of (2) and (3).[8] Secondly, the decoupling of (2) and (3) that occurs when $\pi=0$ also occurs on any three-surface for which the scalar $T\equiv\frac{2}{3}\gamma^{-1/2}\pi$ is independent of position, i.e., for which $\partial T/\partial x^a=0$.[8] This is because if $T=$ const, then (2) says that the trace-free part of π^{ab} is transverse. Hence, *if* $\tilde{\gamma}_{ab}$ *and a matter distribution are specified arbitrarily, and if* π^{ab} *satisfies (2), then the full Riemannian structure of* $T=$ *const surfaces (conformal geometry plus scale factor) is uniquely determined.*

The above results lead to the choice of T and $\tilde{\gamma}_{ab}$ as the independent dynamical coordinates,[9] since they are the quantities that define the field configuration and are freely specifiable in the initial-value problem. (Of course, one must recall that $\gamma^{ab}\delta\tilde{\gamma}_{ab}=0$.) The variable conjugate to T is just $\gamma^{1/2}$, the elementary measure of volume, i.e., of scale. Let us now point out a few of the properties that lead to the identification of T as "time."

The main idea is that the rate of change of T in timelike directions tends to be positive as a consequence of the equations of motion. Let the unit timelike normal field of the spacelike surfaces be denoted $u^\alpha(x)$. With the recognition that $T=\frac{2}{3}\gamma^{-1/2}\pi=-\frac{4}{3}u_{;\alpha}{}^\alpha$, it can be shown from Einstein's vacuum equations that[10]

$$\pounds_u T=\tfrac{1}{4}T^2+\tfrac{3}{4}\sigma_{\mu\nu}\sigma^{\mu\nu}-\tfrac{3}{4}a_{;\lambda}{}^\lambda, \tag{6}$$

where $\pounds_u$ denotes Lie differentiation along u^α, $\sigma^{\mu\nu}$ is the shear of the congruence $u^\alpha(x)$, and a^μ is the four-acceleration $a^\lambda=u_{;\nu}{}^\lambda u^\nu$. We see that for freely falling observers ($a^\lambda=0$) $\pounds_u T\geqslant0$, i.e., T increases with respect to the local standard of proper time. To make the properties of T, which is essentially the volume Hubble parameter, more evident, suppose that we choose a surface on which $T=$ const. Now set $T=t$ (time coordinate) and determine the orthogonal proper time ($N dt$) to the surface $t+dt=$ const. It follows from $\partial T/\partial t=1$ and $a_{;\lambda}{}^\lambda=N^{-1}\nabla^2N$ that the lapse function N must satisfy in vacuum

$$(-\nabla^2+\sigma_{\mu\nu}\sigma^{\mu\nu}+\tfrac{1}{3}T^2)N=\text{const}. \tag{7}$$

In closed empty universes the quantity $\sigma_{\mu\nu}\sigma^{\mu\nu}+\frac{1}{3}T^2$

is strictly positive so that N will exist and be uniquely determined, except at a moment of time symmetry when one must make a transformation on a T of the type described below. Thus T defines a definite slicing of space-time. The presence of matter does not essentially alter these arguments.[8]

The extrinsic scalar time has none of its essential properties changed by transformations of the type $T'=T'(T)$, where $\partial T'/\partial T>0$, and is independent of position on $T=$ const surfaces. Moreover, it continues to increase right through a moment of maximum expansion of the universe and its sign could be used to distinguish the expansion and contraction epochs. Kuchař has pointed out the latter two aspects of extrinsic-type time variables and has discussed some implications concerning the quantization of gravity.[11] The properties of the scalar extrinsic time T contrast with those of intrinsic-type time variables such as Misner's[12] choice $\Omega=-\frac{1}{3}\ln\gamma^{1/2}$. At a moment of maximum expansion, Ω stops its forward flow and begins to run backward. Moreover, it is not a scalar and thus has utility only in the presence of a definite choice of three-dimensional coordinates. However, one can show that $4\mathfrak{L}_u\Omega=T$, so that for homogeneous models Ω and T can define the same family of three-spaces. The use of T as time does *not* depend, though, on any assumptions of homogeneity, nor does it restrict in any way the anisotropy.

In the ADM[6] approach to general relativity one attempts to solve $\mathcal{H}=0$ for the "true" nonvanishing Hamiltonian. With the choice of T as time, the nonvanishing Hamiltonian density becomes the scale factor $\gamma^{1/2}$, so that the full Hamiltonian becomes equal to the volume of the universe. Therefore, *by solving (4) for the conformal factor φ and satisfying the constraints, we are at the same time finding the nonvanishing Hamiltonian of general relativity.*

The action principle of general relativity is based on the invariant Lagrange density $\mathfrak{L}={}^4R(-g)^{1/2}$. It can be decomposed relative to a family of spacelike hypersurfaces with unit normal field u^λ into the form[8]

$$\mathfrak{L}=(-g)^{1/2}(R+K_{ab}K^{ab}-K^2)-2\partial_\lambda(-g)^{1/2}(Ku^\lambda+a^\lambda), \tag{8}$$

where K_{ab} is the extrinsic curvature and $K=\gamma^{ab}K_{ab}$. The first term is the standard Lagrangian of geometrodynamics[13] and will be denoted $\mathfrak{L}_G$. The second term is a pure space-time divergence which may be transformed in the action principle to a boundary term. The boundary integral involving a^λ vanishes identically since $a^\lambda u_\lambda=0$, which shows that the dynamics of gravitation is completely independent of the acceleration of observers.[14] The boundary integral involving Ku^λ does not vanish but plays a fundamental role in the present considerations. To see its meaning, we write, in terms of ADM variables,

$$-2\partial_\lambda(-g)^{1/2}Ku^\lambda=\partial_a\pi N^a-\partial_0\pi, \tag{9}$$

where $N^a=\gamma^{ab}g_{0b}$ is the shift vector and $\partial_0=\partial/\partial t$. The spatial divergence in (9) can be discarded, giving for the action integral

$$S=\int_\Omega\mathfrak{L}\,d^4x=\int_\Omega(\mathfrak{L}_G-\partial_0\pi)\,d^4x=S_G-\int_{\partial\Omega}\pi\,d^3x, \tag{10}$$

where $S_G=\int\mathfrak{L}_G\,d^4x$. We may readily evaluate the action S for dynamical paths, i.e., for solutions of ${}^4R_{\mu\nu}=0$. Since for these paths $\mathfrak{L}=0$, we find

$$0=\int_\Omega\mathfrak{L}\,d^4x=S_G-\int\pi\,d^3x; \tag{11}$$

therefore, as an immediate consequence,[14,15]

$$S_G=\int_{\partial\Omega}\pi\,d^3x. \tag{12}$$

If we choose the bounding spacelike hypersurfaces as $T_2=$ const and $T_1=$ const and recall that $\pi=\frac{3}{2}\gamma^{1/2}T$, we obtain

$$S_G=\tfrac{3}{2}(T_1V_1-T_2V_2), \tag{13}$$

where V is the total volume. Since in the present approach the total volume and total Hamiltonians are equal, we have arrived at a simple expression for the action in terms of H and T. As we indicated above, for T and the conformal geometry ${}^3\tilde{\mathcal{G}}$ given, the initial value equations determine the scale, i.e., H. Thus we arrive at the conclusion that, as a functional, $S_G=S_G[{}^3\tilde{\mathcal{G}},T]$. We expect analogous conclusions to hold in any quantized version of general relativity. Note that the configuration space that one is led to by the initial-value equations is not superspace (the space of Riemannian three-geometries), but "conformal superspace" [the space of which each point is a conformal equivalence class of Riemannian three-geometries]×[the real line] (i.e., the time T).

We can now mention briefly two important results based on recent investigations of others which strongly support the identification of conformal three-geometry and true gravitational degrees of freedom.

First we point to Misner's quantization of the "mix-master universe."[12] The actual degrees of freedom quantized by his procedure refer to the *anisotropy of space*. Clearly, the most general concept of spatial anisotropy is simply the conformal three-geometry. Of course, the mixmaster model is spatially homogeneous and our considerations are by no means limited to that, but in principle these approaches are in very close accord.

Secondly, the work of Brill and Deser[16] on the positivity of gravitational energy for asymptotically flat three-spaces is in consonance with the present approach in its conclusions. Brill and Deser showed that on a maximal hypersurface, for first-order perturbations which obey the constraints, purely conformal variations of the metric $\delta\gamma_{ab} = \lambda\gamma_{ab}$ have no effect on the total energy. Furthermore, if the conformal factor is fixed at spatial infinity, it is unique[7]; thus there exists no such λ. Therefore, the gravitational energy is associated only with the dynamics of the conformal three-geometry. This energy is implicitly defined by the expression $\mathcal{H} = 0$. A study of the solutions of this constraint is under way and is expected to yield a definite answer on the question of the positivity of the energy without resort to perturbation techniques, which are subject to certain drawbacks.[16]

As a final remark, one can see from quite elementary and well known considerations that conformal space geometry is a concept well adapted to the description of dynamical gravitational fields. A weak, plane gravitational wave in the lowest-order approximation only changes the shape, not the volume, of a small cloud of test particles through which it passes, i.e., it initially imparts a slight shearing motion to them. This means that only the conformal geometry associated with the particles is initially affected. A rigorous geometrical expression of this idea is given by the fact that the invariant rate of change of the conformal metric is proportional to the shear: $\pounds_u \tilde{\gamma}_{ab} = 2\gamma^{-1/3}\sigma_{ab}$.

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