Electrical Conductivity in Toroidal Plasma with Trapped Particles

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The electrical conductivity in a toroidal plasma has been calculated, using the exact Fokker-Planck operators for electron-electron and electron-ion collisions. It is found that the nonlinear effects of the electron-electron operator are of order r/R, where r and R are the minor and major radii of a flux surface. The conductivity can be written as $\sigma=\sigma_{\text{Spitzer}}[1-\frac{3}{2}(r/R)^{1/2}+O(r/R)]$.

Recently, Salat[1] and Galeev[2] have attempted to improve on the Lorentzian-model calculations of Hinton and Oberman[3] for the electrical conductivity of a toroidally confined plasma. The improvement has been sought by including a Bhatnagar-Gross-Krook model operator for electron-electron collisions in the calculation. Corrections given by Salat and Galeev seem, however, to differ.

In this paper the problem has been reconsidered utilizing the Fokker-Planck operators for both electron-electron and electron-ion collisions.[4] As in Ref. 3, it is assumed that the magnetic field configuration can be approximated as $B(r,l)=B_0[1+\Delta\cos(\pi l/L)]$, where $\Delta=2r/R$ and $L=4\pi^2R/\iota$ (where r and R are the minor and major radii of a flux surface and ι is the rotational transform). A power-series assumption for the solution of the kinetic equation in Δ fails to satisfy the boundary conditions in the "trapped-particle" region, where the pitch angle is $\theta\leq\pi/2\pm\sqrt{\Delta}$. Therefore, two separate solutions for trapped- and untrapped-particle regions are found which are then matched[5] to yield a uniformly valid first approximation to the distribution function for a parallel electric field. Our main result, in accordance with the conjecture of Ref. 3, is that the electron-electron operator mainly restores the Spitzer conductivity factor and its nonlinear effect due to toroidal geometry is of order Δ. Salat and Galeev, however, have also corrections of order $\sqrt{\Delta}$.

The electron-ion and electron-electron collision operators can be written most conveniently[4] as $C_{ei}=(\nu_{ei}/2)(v_{\text{th}}/v)^3\nabla_v\cdot(v^2\overleftrightarrow{\text{I}}-\vec{v}\vec{v})\cdot\nabla_vF$, and $C_{ee}=\nu_{ee}(v_{\text{th}}{}^3/2n)[8\pi F^2+\nabla_v\nabla_vG(F):\nabla_v\nabla_vF]$, where $G(F)$ is the Rosenbluth potential, i.e., $G(F)=\int F(\vec{v}')|\vec{v}'-\vec{v}|d^3v'$. For ν_{ei} and v_{th} we adopt the definitions given in Ref. 3. Further, we define a mean electron-electron collision frequency as $\nu_{ee}=(n/n_i)\nu_{ei}/Z^2$, where Z is the ionic charge number. We employ the usual[3] coordinate system in velocity space (r,θ,φ) with $\vec{v}=\vec{n}v\cos\theta+v\sin\theta\times(\vec{p}\cos\varphi+\vec{b}\sin\varphi)$. In the limit $E/E_c\sim v_{\text{th}}/L\Omega\ll1$, where $E_c=mv_{\text{th}}\nu_{ei}/e$ and L is a scaling length, we can assume $F\sim F_0+\alpha F_1+\alpha^2F_2+\cdots$, where $\alpha\equiv E/E_c$. The F_i satisfy a set of equations derived by Frieman.[6] A rederivation with some corrections is also given by Daybelge.[7] Specializing to an axisymmetric torus and assuming F_0 to be Maxwellian, the kinetic equation for the averaged F_1, that is $\bar{F}_1\equiv f=(2\pi)^{-1}\int_0^{2\pi}F_1(v,\theta,\varphi)d\varphi$, becomes[3]

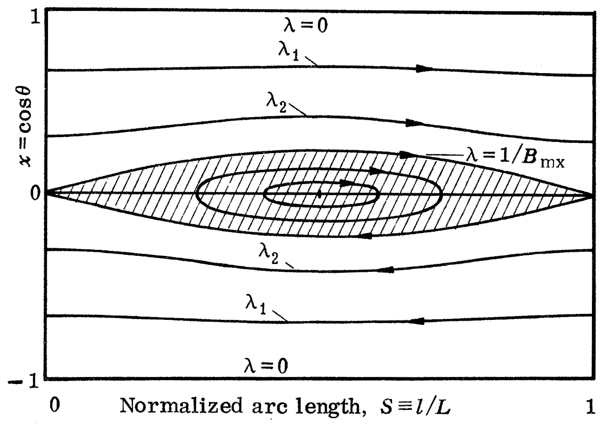

FIG. 1. Trapped- and untrapped-particle regions of the phase space.

$$v\cos\theta\frac{\partial f}{\partial l}+v\frac{\sin\theta}{2B}\frac{\partial B}{\partial l}\frac{\partial f}{\partial\theta}-\bar{C}_{ei}-\bar{C}_{ee}=\frac{e}{m}E_{\parallel}\cos\theta\frac{\partial F_M}{\partial v}-\vec{v}_D\cdot\nabla_rF_M. \quad (1)$$

Here, we are interested in finding a particular solution of (1) for the driving force $E_{\parallel}$. The new operator in (1) can be written as

$$\bar{C}_{ee}=\nu_{ee}v_{\text{th}}{}^3\frac{1}{4\pi n}\int_0^{2\pi}[16\pi F_MF_1+\nabla_v\nabla_vG(F_M):\nabla_v\nabla_vF_1+\nabla_v\nabla_vG(F_1):\nabla_v\nabla_vF_M]d\varphi. \quad (2)$$

The dyads $\nabla_v\nabla_v F_M = g_1(v)\vec{v}\vec{v} + g_2(v)\overleftrightarrow{I}$ and $\nabla_v\nabla_v G(F_M) = h_1(v)\vec{v}\vec{v} + h_2(v)\overleftrightarrow{I}$ are given in Ref. 4. We will adopt the boundary conditions as stated in Ref. 3 and note that the pitch angle θ is related to the ratio λ of magnetic moment to kinetic energy as $\lambda = \sin^2\theta/B(l, \alpha, \beta)$. We also define a new phase-space variable x as $x \equiv \cos\theta = \pm[1-\lambda B(l)]^{1/2}$, which will be used in the sequel.

The shaded area in Fig. 1 is the trapped-particle region. The equation of the separatrix is $\lambda = 1/B_{\max}$ or $x = \pm[\Delta/(1+\Delta)]^{1/2}\sin(\pi l/L)$. For convenience we define $\epsilon = \sqrt{\Delta} = (2r/R)^{1/2}$.

On substituting the assumed variation of the magnetic field in (1) and transforming to the new variables x and $s \equiv l/L$, we find

$$\frac{\partial f}{\partial s} + \epsilon^2\frac{\pi}{2}\frac{\sin 2\pi s}{1+\epsilon^2\cos^2\pi s}\frac{1-x^2}{x}\frac{\partial f}{\partial x} + \frac{v_{\rm th}\eta}{\nu_{ei}vx}(\overline{C}_{ei}+\overline{C}_{ee}) = \eta\frac{E_\parallel}{E_c}\frac{v_{\rm th}^2}{v}\frac{\partial F_M}{\partial v}. \tag{3}$$

Assuming that the mean free path is much longer than the magnetic field period L, we find the parameter $\eta \equiv \nu_{ei}L/v_{\rm th} = \nu_{ei}/\omega_{\rm bounce}$ to be much smaller than unity. Now we want to obtain an asymptotic solution for f to lowest order in both η and ϵ. Although the second term in (3) suggests that f might be expanded in only even powers of ϵ, such an expansion would not be compatible with the conditions in the trapped-particle region, which is of order ϵ. The boundary-layer-like nature of this region necessitates a specialized solution valid in this region to be matched and complemented by the outer solution. Calculation of the higher approximations in η is complicated by the existence of another boundary layer between the trapped- and untrapped-particle regions. Here we will not deal with this problem.

We expand the distribution function f as $f \sim f_0 + \eta f_1 + \cdots$, and obtain from (3) the following:

$$Lf_0 \equiv \frac{\partial f_0}{\partial s} + \epsilon^2\frac{\pi}{2}\frac{\sin 2\pi s}{1+\epsilon^2\cos^2\pi s}\frac{1-x^2}{x}\frac{\partial f_0}{\partial x} = 0, \tag{4}$$

$$Lf_1 + \frac{v_{\rm th}}{\nu_{ei}vx}[\overline{C}_{ei}(f_0) + \overline{C}_{ee}(f_0)] = \frac{E_\parallel(s)}{E_c}\frac{v_{\rm th}^2}{v}\frac{\partial F_M}{\partial v}, \tag{5}$$

where

$$\overline{C}_{ei}(f_0) = \nu_{ei}\left(\frac{v_{\rm th}}{v}\right)^3\frac{1}{2}\left[-2x\frac{\partial f_0}{\partial x} + (1-x^2)\frac{\partial^2 f_0}{\partial x^2}\right], \tag{6}$$

$$\overline{C}_{ee}(f_0) = \nu_{ee}\frac{v_{\rm th}^3}{2n}\left\{16\pi F_M f_0 + \nabla_v\nabla_v\int f(x',\epsilon)|\vec{v}-\vec{v}'|d^3v : \nabla_v\nabla_v F_M + \frac{h_2(1-x^2)}{v^2}\frac{\partial^2 f_0}{\partial x^2}\right.$$
$$\left. + (v^2h_1 + h_2)\frac{\partial^2 f_0}{\partial v^2} + \frac{2h_2}{v}\frac{\partial}{\partial v} - \frac{x}{v}\frac{\partial}{\partial x}\ f_0\right\}. \tag{7}$$

Equation (4) shows that when x is far from the trapped-particle region, f_0 is independent of s to order ϵ^2. Neglecting terms of order ϵ^2, we can eliminate the first term in (5) through integration over s from zero to one, and consideration of condition (b). The result is the familiar Fokker-Planck equation[4] for fully ionized plasma in straight geometry, whose solution can be found by successive approximations.[4,8,9] Here we will quote the familiar result for $(f_0)_{\rm outer}$:

$$(f_0)_{\rm outer} = -\frac{e\langle E_\parallel\rangle}{kT}xvF_M\sum_{r=0}^{N-1}\delta^r(N)S_{3/2}^{(r)}(\mathrm{v}^2). \tag{8}$$

Here, $S_{3/2}^{(r)}$ is a Sonine polynomial of order $\frac{3}{2}$ and degree r, $\mathrm{v}^2 \equiv mv^2/2kT$, and $\langle E_\parallel\rangle \equiv \int_0^1 E_\parallel(s)\,ds$. In the Nth approximation the coefficients $\delta^r(N)$ are found as $\delta^r = (-1)^r 3D_{0r}/2D$ where D denotes the determinant of an $N\times N$ matrix a_{ij}, and D_{0r} is the determinant of the matrix resulting when the 0th row and rth column of a_{ij} is deleted. Using the matrices H_{rs}^{ei} and H_{rs}^{ee} of Ref. 4, we can write $a_{ij} = H_{ij}^{ei} + H_{ij}^{ee}$.

The form of the outer solution (8) indicates that it cannot satisfy the inner-boundary condition (b). In fact it breaks down when $x \sim \epsilon$. Now we introduce a magnified inner variable τ appropriate to the region of trapped particles by setting $\tau = x/\epsilon$ and $f(v,x;\epsilon;\eta) = \Pi(v,\tau;\epsilon;\eta)$. Accordingly, the original equations (4) and (5) are transformed to (expanding again $\Pi = \Pi_0 + \eta\Pi_1 + \cdots$)

$$\hat{L}\Pi_0 \equiv \frac{\partial \Pi_0}{\partial s} + \frac{\pi}{2}\sin 2\pi s\frac{1}{\tau}\frac{\partial \Pi_0}{\partial \tau} = O(\epsilon^2), \tag{9}$$

$$\hat{L}\Pi_1 + \epsilon^{-3}\frac{1}{2\tau}\left(\frac{v_{\rm th}}{v}\right)^4\left(1 + \frac{\nu_{ee}}{\nu_{ei}}\frac{vh_2}{n}\right)\frac{\partial^2\Pi_0}{\partial\tau^2} = O(\epsilon^{-1}). \tag{10}$$

In obtaining (10) we used the fact that application of $(\nabla_v\nabla_v)_{\rm diag}$ on the kernel of the integral operator creates terms of order one and higher (under transformation $\tau = x/\epsilon$). Equation (9) has the characteristics $\xi = \tau^2 + \frac{1}{2}\cos 2\pi s = \text{const}$, which are consistent with (d). By transformation from (s, τ) to (s, ξ), Eq. (9) yields $\hat{L}\Pi_0 \equiv \partial\Pi_0(s, \xi)/\partial s = O(\epsilon^2)$. Thus, to order ϵ^2, Π_0 is independent of s. The curves $\xi = \text{const}$ differ from $\lambda = \text{const}$ only to order ϵ^2. On the separatrix $\xi = \frac{1}{2}$. From (10) we find

$$\frac{\partial\Pi_1}{\partial s} + \epsilon^{-3}\frac{1}{2\tau}\left(\frac{v_{\rm th}}{v}\right)^4\left(1 + \frac{\nu_{ee}vh_2}{n\nu_{ei}}\right)\left(2\frac{\partial\Pi_0}{\partial\xi} + 4\tau^2\frac{\partial^2\Pi_0}{\partial\xi^2}\right) = O(\epsilon^{-1}), \tag{11}$$

where τ stands for $(\xi - \frac{1}{2}\cos 2\pi s)^{1/2}$. The first term of (11) can be annihilated through integration over s, since for untrapped particles Π_1 is periodic in s. On neglecting terms of order ϵ^2 there results $\partial\Pi_0/\partial\xi + 2\xi\partial^2\Pi_0/\partial\xi^2 = 0$. Its solution for untrapped particles is $\Pi_0 = C_1 + C_2\sqrt{\xi}$. The constants C_1 and C_2 are to be found from matching and boundary conditions: Matching with the outer solution for $x \sim 1$ or $\tau \to \infty$ yields

$$C_2 = -\,\text{sgn}(x)\epsilon F_M v\left(\frac{e\langle E_\parallel\rangle}{kT}\right)\sum_{r=0}^{N=1}\delta^r S_{3/2}{}^{(r)}(v^2).$$

On the separatrix the solution for trapped and untrapped particles must be identical. Characteristics in the trapped-particle region show that here Π_0 is an even function of x. However, outside this region Π_0 is odd. Therefore, Π_0 on the separatrix is zero, and $C_1 = -C_2/\sqrt{2}$. Furthermore, it follows from condition (e) that inside the trapped-particle region Π_0 must vanish, as it otherwise would perturb the particle-number density. The final result for f_0, correct to first order in ϵ, is then

$$f_0 = \begin{cases} -\,\text{sgn}(x)\dfrac{eE_\parallel}{kT}vF_M\displaystyle\sum_{r=0}^{N-1}\delta^r S_{3/2}{}^{(r)}(v^2)\left[-\dfrac{\epsilon}{\sqrt{2}} + \left(x^2 + \epsilon^2\dfrac{\cos 2\pi s}{2}\right)^{1/2}\right] & \text{(untrapped particles)} \\ 0 & \text{(trapped particles).} \end{cases} \tag{12}$$

The average velocity of electrons at s is

$$\vec{V}(s) = 4\pi\int_{\epsilon\sin\pi s}^{1} dx\int_0^\infty x f_0(x, v)v^3\,dv\,\vec{n}. \tag{13}$$

We can write the final result for $\vec{V}(s)$ using the orthogonality of Sonine polynomials as

$$\vec{V}(s) = -\frac{e\langle E_\parallel\rangle\,\delta^0}{m}\left[1 - \frac{3}{2\sqrt{2}}\epsilon + O(\epsilon^2)\right]\vec{n}. \tag{14}$$

For comparison with Ref. 3 we note that $3\epsilon/2\sqrt{2} \simeq 1.06\sqrt{\Delta}$. When $\nu_{ei} \simeq \nu_{ee}$, it is found that $\delta_{\rm Sp}{}^0 = 1.975 \times (3\pi^{1/2}/4\nu_{ei})$. In the Lorentzian limit, i.e., $\nu_{ee} \ll \nu_{ei}$, $\delta_{\rm Lo}{}^0 = 8/\pi^{1/2}\nu_{ei}$. Finally, the parallel electrical conductivity $\sigma \equiv j/\langle E_\parallel\rangle$, where j is the current density driven by the parallel electric field, can be written for $Z = 1$ and $n = n_i$ as

$$\sigma = \sigma_{\rm Sp}\left[1 - \frac{3}{2\sqrt{2}}\Delta^{1/2} + O(\Delta)\right], \tag{15}$$

where $\sigma_{\rm Sp} \equiv 1.975(3\pi^{1/2}/4)ne^2/m\nu_{ei}$ is the Spitzer conductivity.

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Direct Observation of Spin-State Mixing in Superconductors*

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We observe a peak in the tunneling conductance between two very thin superconducting aluminum films in an intense parallel magnetic field at a voltage $V=(\Delta_1+\Delta_2-2\mu H)/e$. The magnetic field dependence of this peak identifies it as that predicted by Engler and Fulde for spin-state mixing in superconductors. The spin-orbit scattering rate $b=\hbar/3\Delta\tau_{s.o.}$ obtained from this measurement is approximately 0.1.

This experiment reports the direct detection of the effect of spin-orbit scattering on the spin density of states of a superconductor. The quasiparticle energy states in superconducting Al are split in a high magnetic field by the interaction of the field with the spin magnetic moment of the quasiparticles. This splitting was recently demonstrated[1] by conductance measurements on Al-Al_2O_3-Ag tunnel junctions (Al thickness ≈ 50 Å). The measured conductances could be analyzed surprisingly well by assuming simply that the BCS[2] density of states was split into spin-up and spin-down parts displaced in energy by $\pm\mu H$ (μ being the electron magnetic moment). The success of this analysis implied that there were no significant interactions to mix the spin states. However, measurements[3,4] of the critical magnetic field H_c of such films indicated that spin-orbit scattering, though small in Al, is not negligible. In fact, theoretical fitting[5] of values of $H_c(T)$ gave a spin-orbit scattering parameter $b=\hbar/3\Delta\tau_{s.o.}\approx 0.2$, large enough to have had a significant effect on the tunneling density of states. Here 2Δ is the energy gap of the superconductor and $\tau_{s.o.}$ is the spin-orbit scattering time.

Recently Engler and Fulde[6] succeeded in calculating the density of states of a thin superconductor in a high magnetic field for various values of b. For $b=0$ and a high magnetic field, $H=0.6(\Delta/\mu)$, Fig. 1(a) shows the calculated density of states $N(E/\Delta)$. As expected, the density of states for each spin direction is just half of the BCS density of states and is shifted in energy by $+\mu H$ for spin up and by $-\mu H$ for spin down. Figure 1(b) shows the interesting theoretical result of increasing b. Here the field is the same but $b=0.2$ and the calculated behavior is qualitatively different, the spin states being partially mixed. The states near $E/\Delta=1-\mu H/\Delta$, for example, are no longer only spin down, as there is also a small spin-up peak. As b increases, the spin mixing increases, and the peaks move closer together and become more nearly equal in magnitude. By the time $b=5.0$, spin is no longer a good quantum number, and $N(E/\Delta)$ has only a single peak, which is naturally independent of H and approaches closely the single, unsplit, BCS

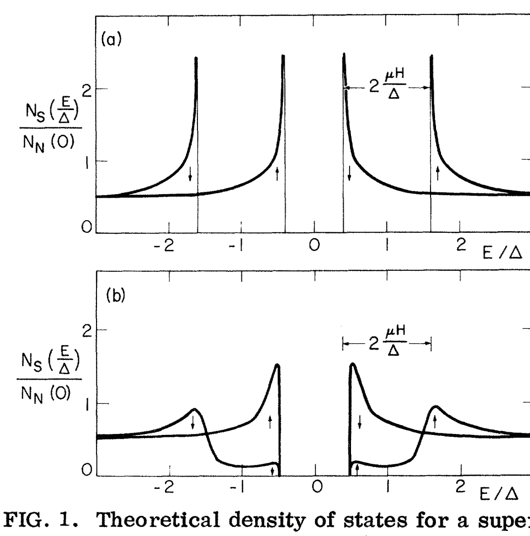

FIG. 1. Theoretical density of states for a superconcuctor in a magnetic field $H=0.6\Delta/\mu$. (a) With $b=0$, $N_s(E/\Delta)$ splits into two BCS-like curves, one for each spin direction shifted by $\pm\mu H$. (b) With some spin-orbit scattering, $b=0.2$, the spin states are mixed with some spin-up states found near the energy of the spin-down peak.