Exact Ground State of Some One-Dimensional N-Body Systems with Inverse ("Coulomb-Like") and Inverse-Square ("Centrifugal") Pair Potentials

F. Calogero
Istituto di Fisica dell'Università di Roma, Rome, Italy, and Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Rome, Italy

and

C. Marchioro
Istituto di Matematica dell'Università di Roma, Rome, Italy
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We exhibit the exact ground state and the corresponding binding energy of the system composed of N (nonidentical) quantum-mechanical particles interacting in one dimension via the potential $V_{ij}(x_i-x_j)=g(x_i-x_j)^{-2}+(f_i-f_j)(x_i-x_j)^{-1}$. We discuss some special choices of the constants f_i and g that yield examples of possible physical interest.

Exactly solvable problems are interesting mathematically and may be useful as approximate schematizations of real models or to test approximation techniques. But even in one dimension, the number of exactly solvable quantum-mechanical many-body models with two-body forces is scarce; in fact, besides the trivial case with oscillator forces, only two solvable examples are known: the system of N equal particles interacting pairwise via δ-function potentials[1] and that with inverse-square ("centrifugal") potentials (possibly also with oscillator forces).[2] We present here another class of models whose exact ground-state wave function, and the corresponding binding energy, can be explicitly exhibited.

We consider the system of N nonrelativistic quantum-mechanical particles of equal mass m interacting in one dimension via the pair potential given in the Abstract. The Hamiltonian of the problem is

$$H=-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+\sum_{i>j=1}^{N}\left[g(x_i-x_j)^{-2}+(f_i-f_j)(x_i-x_j)^{-1}\right]. \quad (1)$$

We assume hereafter that $g>-\hbar^2/4m$, to prevent two-body collapse,[2] and we restrict attention to the sector of configuration space characterized by the restrictions

$$x_1\leq x_2\leq x_3\leq\cdots\leq x_N. \quad (2)$$

It is sufficient to consider this sector because the singular "centrifugal" interaction forbids the particles from overtaking each other, so that their ordering is a constant of the motion.[2]

The ground-state wave function of this system (in the c.m. frame) is

$$\psi(x_1,x_2,\cdots,x_N)=Cz^{\alpha}\exp\left[\sum_{i=1}^{N}q_i x_i\right], \quad (3)$$

with

$$z=\prod_{i>j=1}^{N}(x_i-x_j), \quad (4)$$

$$\alpha=\tfrac{1}{2}[1+(1+4mg\hbar^{-2})^{1/2}], \quad (5)$$

$$q_i=m\hbar^{-2}\alpha^{-1}\left(f_i-N^{-1}\sum_{j=1}^{N}f_j\right). \quad (6)$$

C is a normalization constant (we assume here that ψ is normalizable; necessary and sufficient conditions for this are discussed below).

The energy of this state is

$$E=-(\hbar^2/2m)\sum_{i=1}^{N}q_i^2. \quad (7)$$

The proof of these assertions can be performed verifying that ψ and E satisfy the Schrödinger equation

$$H\psi=E\psi. \quad (8)$$

The computation is essentially trivial if the following identities are used[3]:

$$\sum_{i=1}^{N}\partial^2 z/\partial x_i^2=0, \quad (9a)$$

$$z^{-2}\sum_{i=1}^{N}(\partial z/\partial x_i)^2=2\sum_{i>j=1}^{N}(x_i-x_j)^{-2}. \quad (9b)$$

The fact that the eigenfunction (3) corresponds to the ground state is implied by the remark that ψ has no zeros besides those due to the presence of the (singular) "centrifugal" potential that forces ψ to vanish whenever $x_i = x_j$.

It is easy to verify (for instance going over to the $N-1$ independent variables $y_i = x_{i+1} - x_i$) that the necessary and sufficient conditions for ψ to be normalizable are

$$\sum_{i=1}^{j} q_i > 0, \quad j = 1, 2, 3, \cdots, N-1, \tag{10a}$$

or, equivalently,

$$\sum_{i=j}^{N} q_i < 0, \quad j = 2, 3, 4, \cdots, N. \tag{10b}$$

Hereafter we consider only models that satisfy these conditions.

It is plausible to conjecture that if the constants f_i, which determine the strength of the "Coulomb-like" part of the potential, violate the conditions implied by Eqs. (10) and (6), the system does not possess an N-body bound state. Note that these conditions of boundedness are independent of the value of the coupling constant g characterizing the strength of the centrifugal interaction that plays essentially the role of a scale parameter.

We have implicitly assumed that the "centrifugal" potential is present, i.e., that g does not vanish. All results remain valid (with $g=0$, $\alpha = 1$) if the "centrifugal" potential is replaced by an infinitely repulsive zero-range potential or, equivalently, by a boundary condition forcing the wave function to vanish whenever the coordinates of two particles coincide.

Special choices of the constants f_i characterizing the "Coulomb-like" part of the interaction are worth considering. We mention four representative examples.

(i) Assume that N is even. Let

$$f_i = \tfrac{1}{2}(-)^{i+1} f[1 + \epsilon(\delta_{i1} + \delta_{iN})], \quad f > 0, \quad \epsilon > 0. \tag{11}$$

For $\epsilon = 0$, the interaction may be described as follows: Odd-numbered particles do not interact between themselves, they attract the even-numbered particles to their right via the potential $-f/|x_i - x_j|$, and they repel the even-numbered particles to their left via the potential $f/|x_i - x_j|$; even-numbered particles behave accordingly, i.e., they do not interact between themselves, they attract the odd numbered particles to their left, and they repel the odd-numbered particles to their right. However, for $\epsilon = 0$, the choice (11) violates the boundedness conditions (10) although it satisfies them for any positive ϵ. The presence of ϵ modifies only the interaction of the first and last particles with all the others, increasing the overall attraction so as to bind the N-body bound state (it is easy to convince oneself that the system described above, corresponding to $\epsilon = 0$, tends to break up into n two-body clusters).

Note that this system exhibits saturation, for its ground-state energy is

$$E = \tfrac{1}{8} m\hbar^{-2}\alpha^{-2} f^2 (N + 4\epsilon + 2\epsilon^2). \tag{12}$$

(ii) Let

$$f_i = -fi, \quad f > 0. \tag{13}$$

Then the interaction is attractive for every pair, and its strength is larger among particles that are farther apart:

$$V_{ij}(x_i - x_j) = -f|i-j|/|x_i - x_j|. \tag{14}$$

The ground-state energy is

$$E = -\tfrac{1}{24} m\hbar^{-2}\alpha^{-2} f^2 N(N+1)(N-1). \tag{15}$$

(iii) Let

$$f_i = f/i, \quad f > 0, \tag{16}$$

so that the interaction is attractive for every pair:

$$V_{ij}(x_i - x_j) = -f|i-j|/ij|x_i - x_j|. \tag{17}$$

The ground-state energy is then

$$E = -\tfrac{1}{2} m\hbar^{-2}\alpha^{-2} f^2 \Big[\sum_{i=1}^{N} i^{-2} - N^{-1}\Big(\sum_{i=1}^{N} i^{-1}\Big)^2\Big], \tag{18a}$$

so that, at large N,

$$E = -(\pi^2/12) m\hbar^{-2}\alpha^{-2} f^2 [1 + O((\ln N)^2/N)]. \tag{18b}$$

It is remarkable that in this case, as N diverges, the ground-state energy tends to a finite limit, so that the binding energy per particle vanishes asymptotically. This is of course due to the fact that additional particles interact more and more weakly.

(iv) Let $N = 3$ and

$$f_1 = -\tfrac{1}{3} e_2(2e_2 + e_3), \tag{19a}$$

$$f_2 = \tfrac{1}{3} e_2(e_1 - e_3), \tag{19b}$$

$$f_3 = \tfrac{1}{3} e_2(e_1 + e_3), \tag{19c}$$

with

$$e_2 = e_1 e_3/(e_1 + e_3), \tag{20}$$

e_1 and e_3 arbitrary. Then the interaction becomes

$$\sum_{i>j=1}^{3} e_i e_j / |x_i - x_j|, \tag{21}$$

i.e., it corresponds to the Coulomb interaction of three particles of charge e_1, e_2, and e_3. It is then easily seen that the boundedness conditions (10) yield

$$e_1 e_3 < 0 \tag{22a}$$

and

$$|e_1| > 2|e_3| \text{ or } |e_3| > 2|e_1|. \tag{22b}$$

Thus, if these conditions are satisfied, the ground-state of the system is the three-body bound state described by the function ψ of Eq. (3), and the corresponding energy is

$$E = -\tfrac{1}{3} m\hbar^{-2}\alpha^{-2} e_1^2 e_3^2 \frac{(e_1^2 + e_1 e_3 + e_3^2)}{(e_1 + e_3)^2}. \tag{23}$$

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Two Experimental Inputs to the λ-Point Helium Paradox*

John R. Pellam
University of California, Irvine, California 92664
(Received 17 May 1971)

Interpretations of the "λ-point paradox" are narrowed significantly by recent experimental findings. Penney's proposal that the paradox originates in effects of room-temperature radiation falling on the disk is disproved. Further, by rotating the disk through the helium rather than rotating the liquid past the disk, the paradox is found to be dependent only on relative rotation. This invariance to absolute rotation of helium evidently eliminates any role of the rotational properties *per se* of the liquid.

Although by no means offering a complete resolution to the λ-point rotation paradox, the two experimental results reported here do greatly narrow the scope of the problem. One should be reminded that the rotation paradox[1,2] concerns the observed temperature dependence of torque exerted by rotating liquid helium on a Rayleigh disk, in particular the decrease of this torque toward zero as the temperature is increased toward the λ point.

One of the present findings eliminates the effects of radiation as recently proposed to explain the paradox. The other reveals that λ-point phenomenon is in fact rooted in other than the rotation properties *per se* of liquid helium.

(1) *Penney's radiation hypothesis.*—Recently Penney[3] suggested that the rotation paradox might be attributed to the action of room-temperature radiation falling upon the disk. The heating generated over the surface of the disk by such radiation would set up internal convection currents between normal fluid and superfluid, and these in turn would modify the individual fluid flow patterns about the disk. According to Penney's calculations, the process would result in decreased net torque exerted upon the disk similar in behavior to the observed effect. This includes the total disappearance of torque at the λ point.

Perhaps the least convincing aspect of the argument is the apparent strictness placed upon the amount of radiation incident upon the disk. Rather than comprising a "saturation" effect (and thus some minimum threshhold), the process requires a *specific level* of radiation to suppress the torque at the λ point. Immediately the explanation hangs upon the fortuitous values of room-temperature radiation balanced against other factors.

Clearly a test by direct experiment was required. A helium-temperature "radiation envelope" was provided for the test region by painting the outer surface of the original precision glass rotor and the supporting glass tube with Aquadag. Thus, except for two small "windows" left in the Aquadag painting as observation apertures, the rotating liquid-helium sample and suspended disk probe were completely shielded from all but liquid-helium-temperature radiation. Such a geometry permitted the original experimental method of observing disk behavior by measuring deflections of a beam of light reflected from the disk surface. (See Ref. 2, Fig. 1 for the basic system to which Aquadag shielding was applied.)

One small circular hole was provided in the