It is a trivial matter to extend the formalism to a nucleus of charge Z and an atom of Z electrons. However, one can show then that the lowest-order interaction term represents the complete ionization of the atom into Z electrons and a nucleus, and is suitable only if ionization in unlikely. Consequently, it is necessary to introduce the state of the ions with charges $Z-1$, $Z-2$, $\cdots$, Z. This is readily accomplished by modifying Eq. (5). Now I_B becomes the projection operator onto the states of the Z-electron atom. The product of the identities $I(M+2, M+2')\cdots I(N, N')$ itself is decomposed according to Eq. (5): $I_B\cdots I_F I\cdots I$, where I_B and I_F now project onto states of the ion with charge $Z-1$. The product $I\cdots I$ in the above is again decomposed according to (5) where now the projection operator is that for the states of the ion with the charge $Z-2$. One continues this process until the nucleus of charge Z and electrons are left or when the rare-gas core is reached. The physical states will now contain at most one quantum of each of the unbound composites and the Hamiltonian will contain terms corresponding to single ionization, double ionization, etc., of the atom and of the ions. Extension to molecular aggregates and to negative ions is trivial.

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Gravitational Waves in Closed Universes

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New boundary conditions on the Einstein-Rosen-Bondi gravitational-wave metrics yield closed inhomogeneous universes which solve Einstein's vacuum field equations exactly. Space sections of these universes have either the three-sphere topology S^3 or the wormhole (hypertorus) topology $S^1\otimes S^2$.

By means of a change of boundary conditions, the Einstein-Rosen-Bondi analysis of cylindrical and plane waves[1] may be used to construct gravitational waves in closed universes. Spacelike sections of these universes have either the three-sphere topology S^3 which is familiar from the Friedman universe or the hypertorus topology $S^1\otimes S^2$ which may be imposed upon the Kantowski-Sachs universe.[2] Exact solutions to Einstein's field equations are obtained in the absence of matter and nongravitational fields. These solutions provide a new type of cosmological model which is dominated by homogeneity-breaking, coherent gravitational radiation.[3] They also provide an important new "theoretical laboratory" for testing and exploring recent formal developments in general relativity.[4]

An Einstein-Rosen-Bondi space-time will be defined to be one which possesses two mutually orthogonal, hypersurface orthogonal, spacelike Killing vector fields. It is well known that any such space-time has a metric which may be put into the form

$$ds^2 = L^2[e^{2a}(d\theta^2 - dt^2) + R(B^{-1}e^{2W}d\sigma^2 + Be^{-2W}d\delta^2)], \quad (1)$$

where L is a constant length; t, θ, σ, and δ are dimensionless coordinates; and a, W, R, and B are dimensionless functions of θ and t alone.[5] This form of the metric does not fix the coordinates θ and t completely. In terms of the advanced and retarded coordinates $v\equiv t+\theta$ and $u\equiv t-\theta$, the most general coordinate transformations which preserve the form of the metric are

$$u=F(\tilde{u}), \quad v=G(\tilde{v}), \quad (2)$$

where F and G are arbitrary functions of one variable.[5] In order to write Einstein's equations for this family of space-times, it is convenient to denote derivatives with respect to v and u by the subscripts + and −, respectively, and to define the function

$$\psi \equiv W - \tfrac{1}{2}\ln B. \quad (3)$$

The independent Einstein field equations are then[6]

$$R_+ a_+ = R\psi_+^{\,2} + \tfrac{1}{2}R_{++} - \tfrac{1}{4}R(R_+/R)^2, \quad (4)$$

$$R_- a_- = R\psi_-^{\,2} + \tfrac{1}{2}R_{--} - \tfrac{1}{4}R(R_-/R)^2, \quad (5)$$

$$(\partial/\partial\theta)(R\partial\psi/\partial\theta) - (\partial/\partial t)(R\partial\psi/\partial t) = 0, \quad (6)$$

$$\partial^2 R/\partial\theta^2 - \partial^2 R/\partial t^2 = 0. \quad (7)$$

Equation (7) is the key to constructing closed universes. The general solution to this one-dimensional wave equation is $R = J(u) + K(v)$, where J and K are adjustable functions. The allowed coordinate transformations given by Eq. (2) can bring R into the form $R = J(F(u)) + K(G(v))$. Thus, for a given space-time, the functional form of $R(u,v)$ is highly dependent on the choice of coordinates. However, the allowed transformations cannot alter the light cones of a space-time. Thus, allowed coordinate transformations cannot alter the spacelike, lightlike, or timelike character of the gradient $R_{,\alpha}$ which is therefore an invariant feature of the spacetime.[7] More important for the purpose of constructing solutions with a particular global topology is the fact that allowed coordinate transformations cannot alter the topology of the level surfaces of R and the way in which these surfaces are constructed from spacelike, timelike, and lightlike segments. In particular, the solution

$$R = \sin\theta \sin t \quad (8)$$

is distinct in its global structure from the solutions $R = \theta$ and $R = t$, which lead to the usual Einstein-Rosen cylindrical waves, and also from the solution $R = u^2$, which leads to the Bondi plane-wave metrics.

A space-time described by Eq. (8) can be regular only on a coordinate patch which is bounded by initial and final "collapse" singularities at $t=0$ and $t=\pi$ and by apparent singularities at $\theta=0$ and $\theta=\pi$. Within this square in the (θ, t) plane all three classes of wave metrics ($R_{,\alpha}$ spacelike, lightlike, and timelike) are present and join one another at the lightlike surfaces $u=0$ and $v=\pi$, which form the diagonals of the square. Away from these diagonals, R_+ and R_- are nonzero so that Eqs. (4) and (5) may be solved for a_+ and a_- and integrated directly. However, along the diagonal $v=\pi$, one has $R_+=0$ and Eq. (4) becomes

$$\psi_+(u,\pi) = \pm[2\cos(u/2)]^{-1}, \quad (9)$$

and along the diagonal $u=0$, one has $R_-=0$ and Eq. (5) becomes

$$\psi_-(0,v) = \pm[2\sin(v/2)]^{-1}. \quad (10)$$

When Eq. (6) is written in terms of u and v and evaluated along the diagonals, one finds that it reduces to first-order ordinary differential equations for the normal derivatives of ψ and that Eqs. (9) and (10) are solutions. Thus, the constraints (9) and (10) are propagated in time by Einstein's equations, and it is sufficient to impose them at just one time. It is convenient to impose them at $t=\pi/2$, the time when the diagonals cross at $\theta=\pi/2$. At $t=\theta=\pi/2$, one requires that either

$$\partial\psi/\partial\theta = 0 \text{ and } \partial\psi/\partial t = \pm 1, \quad (11)$$

or

$$\partial\psi/\partial\theta = \pm 1 \text{ and } \partial\psi/\partial t = 0. \quad (12)$$

Each possible pair of constraints yields a distinct family of solutions although some families may be related by time reversal and parity.

From Eqs. (1) and (3) it is clear that W and B are redundant functions. One is free to require W to be one of the *regular* solutions

$$W = \textstyle\sum_l [A_l P_l(\cos t) + C_l Q_l(\cos t)] P_l(\cos\theta) \quad (13)$$

to the wave equation (6) where P_l and Q_l are Legendre functions of the first and second kind and A_l and C_l are adjustable constant coefficients. The function $\ln B$ will then be an irregular solution with logarithmic singularities at $\theta=0$ and $\theta=\pi$. The additional requirement that Eq. (1) describe a closed universe implies further that B should be proportional to either θ or θ^{-1} near $\theta=0$ and to either $\pi-\theta$ or $(\pi-\theta)^{-1}$ near $\theta=\pi$. Two solutions for B which satisfy these requirements are

$$B = \sin\theta \sin t \quad (14)$$

and

$$B = \tan(\theta/2). \quad (15)$$

If Eq. (14) is adopted as the solution for B, then the space-time metric becomes

$$ds^2 = L^2 \sin^2 t [e^{2(\gamma - w)}(d\theta^2 - dt^2) + e^{-2w}\sin^2\theta\, d\delta^2] + L^2 e^{2w} d\sigma^2, \quad (16)$$

where γ is related to the function a by

$$\gamma \equiv a + W - \ln\sin t. \quad (17)$$

This metric describes a closed universe with space sections of topology $S^1 \otimes S^2$ provided that σ and δ are regarded as angle coordinates with period 2π. The metric is regular except for initial and final singularities at $t=0$ and $t=\pi$ and possible conical singularities at $\theta=0$ and $\theta=\pi$. To avoid conicality, it is sufficient to require $\gamma=\partial\gamma/\partial\theta=0$ at $\theta=0$ and $\theta=\pi$. From the Einstein equations (4) and (5), one finds that $\partial\gamma/\partial\theta=0$ and $\partial\gamma/\partial t=\cot t$ whenever $\theta=0$ or $\theta=\pi$. Thus, conicality can be avoided by requiring only $\gamma(0,t)=\gamma(\pi,t)=\ln\sin t$; and furthermore, these conditions need only be imposed at a single time t. Equations (4) and (5) may be integrated whenever Eq. (6) is satisfied. The unique solution for γ which incorporates regularity at $\theta=0$ is

$$\gamma=\ln\sin t+\int_0^\theta dy\,\sin y\{2\sin t[W_+{}^2/\sin(t+y)+W_-{}^2/\sin(t-y)]-\cos y(\sin^2 t-\sin^2 y)^{-1}\}. \tag{18}$$

It is convenient to impose the remaining regularity condition, $\gamma(\pi,t)=0$, at $t=\pi/2$. The result is the quadratic integral constraint,

$$\int_0^\pi d\theta\,\tan\theta[(\partial W/\partial t)^2+(\partial W/\partial\theta)^2-1]|_{t=\pi/2}=0, \tag{19}$$

which may be regarded as a single quadratic equation restricting the infinite set of soefficients A_l and C_l which appear in the solution [Eq. (13)] for W. These coefficients are also restricted by two linear equations arising from Eqs. (11) and (12). For the present choice of B, these constraints are either

$$\partial W/\partial\theta|_{\theta=t=\pi/2}=0 \text{ and } \partial W/\partial t|_{\theta=t=\pi/2}=\pm 1, \tag{20}$$

or

$$\partial W/\partial\theta|_{\theta=t=\pi/2}=\pm 1 \text{ and } \partial W/\partial t|_{\theta=t=\pi/2}=0. \tag{21}$$

Notice that these linear constraints guarantee that Eqs. (9) and (10) are satisfied which, in turn, guarantee that the poles in the integrands of Eqs. (18) and (19) vanish so that the integrals will always exist.

Equations (20) and (21) correspond to two distinct families of "wormhole universes". The arbitrary signs in the equations merely correspond to the possibility of time-reversed and parity-reflected solutions. To obtain a particular solution, one finds coefficients A_l and C_l which satisfy the appropriate pair of linear algebraic equations as well as the quadratic equation (19), and then performs the integral in Eq. (18) to find γ. For example, if Eq. (20) is chosen, then there is a homogeneous wormhole universe which is obtained by letting $C_0=\pm 1$ be the only nonvanishing coefficient. The result is a form of the Kantowski-Sachs metric.[2] If Eq. (21) is chosen, then time-symmetric solutions are possible, and the simplest one is $C_1=\pm 1$ with all other coefficients zero. This family of solutions includes the time-symmetric wormhole universes discussed by Lindquist.[8]

Now consider the consequences of adopting Eq. (15) for B. The resulting space-time metrics are

$$ds^2=\tfrac{1}{2}L^2\sin t[e^{2\gamma}(d\theta^2-dt^2)+4e^{2W}\cos^2(\theta/2)d\sigma^2+4e^{-2W}\sin^2(\theta/2)d\delta^2], \tag{22}$$

where Eq. (17) has been replaced by

$$\gamma\equiv a-\tfrac{1}{2}\ln(\tfrac{1}{2}\sin t). \tag{23}$$

The spacelike part of this metric is a distortion of the three-sphere metric

$$dl^2=d\theta^2+4\cos^2(\theta/2)d\sigma^2+4\sin^2(\theta/2)d\delta^2,$$

where θ, $\frac{1}{2}(\sigma+\delta)$, and $\frac{1}{2}(\sigma-\delta)$ are the Euler angle coordinates.[9] The coordinates σ and δ are angles with period 2π just as in the $S^1\otimes S^2$ case. Regularity at $\theta=0$ and $\theta=\pi$ requires that $\gamma(0,t)=-W(0,t)$ and $\gamma(\pi,t)=W(\pi,t)$. The integral for γ is found to be

$$\gamma=-W(0,t)+\int_0^\theta dy\left\{2R\left[\frac{\psi_+{}^2}{\sin\tilde{v}}+\frac{\psi_-{}^2}{\sin\tilde{u}}\right]-\tfrac{1}{4}\cot y\,\frac{\sin^2 t+3\sin^2 y}{\sin^2 t-\sin^2 y}\right\}, \tag{24}$$

where

$$\psi_\pm\equiv W_\pm\mp(4\sin y)^{-1},\quad \tilde{u}\equiv t-y,\quad \tilde{v}\equiv t+y,$$

and the function W is given by Eq. (13) as before, but with the coefficients A_l and C_l restricted by the quadratic equation

$$\int_0^\theta \frac{d\theta}{\cos\theta}\left\{\sin\theta\left[\left(\frac{\partial W}{\partial t}\right)^2+\left(\frac{\partial W}{\partial\theta}\right)^2-\tfrac{3}{4}\right]-\frac{\partial W}{\partial\theta}\right\}=W(\pi,\tfrac{1}{2}\pi)+W(0,\tfrac{1}{2}\pi) \quad (25)$$

at $t=\pi/2$, and by one of the following three pairs of linear constraints:

$$\partial W/\partial t=0,\quad \partial W/\partial\theta=\tfrac{3}{2}; \quad (26)$$

$$\partial W/\partial t=0,\quad \partial W/\partial\theta=-\tfrac{1}{2}; \quad (27)$$

$$\partial W/\partial t=\pm 1,\quad \partial W/\partial\theta=\tfrac{1}{2} \quad (28)$$

at $\theta=t=\pi/2$. Each pair of linear constraints leads to a distinct family of three-sphere universes. Unlike the wormhole universes, these empty, inhomogeneous generalizations of the Friedman universe do not seem to include any familiar solutions. In particular they do not include the Taub universe.

By replacing the globally regular coordinates θ and t by $R\equiv\sin\theta\sin t$ and $T\equiv\cos\theta\cos t$, one discovers that the metrics discussed here may be written in the form

$$ds^2=e^{(\Gamma-w)}(dR^2-dT^2)+e^w dz^2+R^2e^{-w}d\varphi^2 \quad (29)$$

for R a spacelike coordinate and in the form

$$ds^2=e^{(\Gamma-w)}(dT^2-dR^2)+e^w dz^2+R^2e^{-w}d\varphi^2, \quad (30)$$

where R is timelike. The function w is related to W and B while Γ is related to a or γ. The coordinates z and φ are identified with σ and δ, but the exact nature of the identification (e.g., $z=\sigma$, $z=\delta$, $z=-\sigma$, etc.) may differ from one (R,T) coordinate patch to another. Within the (θ,t) square there are four separate patches within which the Einstein-Rosen coordinates R and T are regular. Equation (29) describes the metric in the patches $(u>0, v<\pi)$ and $(u<0, v>\pi)$ while Eq. (30) is appropriate in the patches $(u<0, v<\pi)$ and $(u>0, v>\pi)$. These metrics appear singular (w has infinite derivatives) along the surfaces $u=0$ and $v=\pi$ where they meet. Thus, with the proper matching conditions [(essentially Eqs. (19)–(21) or (25)–(28)] four Einstein-Rosen cylindrical-wave metrics can be joined to produce a space-regular closed universe which evolves from a singular "big bang" to a final collapse.

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