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Does the Light Cone Dominate the Asymptotic Behavior of Vertex Functions and Scattering Amplitudes?*

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The hypothesis of leading light-cone singularity dominance is examined for vertex functions in the so-called A limit with $\omega = 1$ and for scattering amplitudes in the Regge limit. It is shown that, in the field expansion, terms less singular on the light cone must be as important asymptotically as the more singular terms, unless the asymptotic behavior changes abruptly when an external leg goes slightly off the mass shell.

Recently, Brandt and Preparata[1] applied a light-cone analysis to the study of the vertex functions

$$\tilde{M}(q\cdot p, q^2) = \int d^4x\, e^{-iq\cdot x}\langle 0|T[A(x)B(0)]|p\rangle \tag{1}$$

in the limit $\nu = q\cdot p \to -\infty$, and $\omega = -q^2/2\nu$ fixed (A limit). The basic set of rules that have been developed for such an analysis is that the leading singularity on the light cone $x^2 = 0$ dominates this asymptotic behavior, and that the nature of such singularities can be determined from the light-cone expansion

$$A(x)B(0) \underset{x^2\to 0}{\sim} (x^2 - i\epsilon x_0)^{-r}\sum_n x^{\alpha_1}\cdots x^{\alpha_n}\,\mathcal{O}_{\alpha_1\cdots\alpha_n}(0) \tag{2}$$

with $2r = \dim A + \dim B - N$, $\dim \mathcal{O}_{\alpha_1\cdots\alpha_n} = N + n$. The assignment of dimensions and the availability of local operators $\mathcal{O}_{\alpha_1\cdots\alpha_n}(0)$ with the lowest possible value of N are to be extracted from some definite model like the gluon-quark model, discounting the possibility of anomalous dimensions. Furthermore, in the matrix element

$$\sum_n \langle 0|x^{\alpha_1}\cdots x^{\alpha_n}\,\mathcal{O}_{\alpha_1\cdots\alpha_n}(0)|p\rangle = h(x\cdot p, 0) + \sum_{n=1}^{\infty}\frac{(x^2)^n}{n!}h^{(n)}(x\cdot p, 0),$$

the first term $h(x\cdot p, 0)$ leads to the strongest singularity on the light cone, and is supposed to dominate the integral in (1) in the A limit. From this it was inferred that

$$\tilde{M}(q\cdot p, q^2) \xrightarrow[A\ \text{limit}]{} \nu^{r-2}\int_0^\infty \lambda^{-r+1}h(\lambda, 0)e^{i\omega\lambda}d\lambda \tag{3}$$

(where we have chosen units such that $p^2 = 1$). The authors of Ref. 1 assume that for $\omega \to 1$, corresponding to fixed $(p+q)^2 = k^2$, the integral on the right-hand side of Eq. (3) converges for large λ, and they attribute this to the composite nature of the particle involved. This set of rules has also been applied to a variety of other physical problems with interesting results.

In this note we wish to emphasize that leading light-cone singularity dominance is *a priori* much less compelling in vertex functions with fixed external momentum squared for the two of the legs, with $q^2 \to -\infty$, than in the case of $\omega \neq 1$, $q^2 \to -\infty$, or in scattering amplitudes, on or off the mass shell, in the Regge limit. We further stress that leading light-cone singularity dominance must fail if the asymptotic behavior does not change abruptly when an external leg goes slightly off the mass shell; however, the rules of leading light-cone singularity dominance, in the sense made more specific below, can still be valid if such an abrupt change does occur.

The first reason that light-cone dominance may fail for a particular value of ω is simply that the asymptotic behavior of a multidimensional Fourier integral in momentum space is not always controlled by the singularities of the integrand in coordinate space. Let us denote the integrand of Eq. (1) by $\langle 0|TA(x)B(0)|p\rangle \equiv M(x\cdot p, x^2)$, and take the example

$$M(x\cdot p,x^2)$$
$$=(x^2-i\epsilon)^{-1}u(x\cdot p)+e^{-ip\cdot x}x^2\Delta_F(x^2,m^2),$$

where $u(\lambda)\to$ const as $\lambda\to\infty$. The second term is continuous across the light cone, and yet its Fourier transform approaches a constant as $\nu\to\infty$ with k^2 fixed, dominating the contribution from the singular term.[2] Note that in the A limit with $\omega\neq 1$, $k^2\sim\nu$ as $\nu\to\infty$, and the Fourier transform of the second term approaches ν^{-3} consistent with the rules—hence our emphasis that light-cone dominance may fail for some particular values of ω, in this example for $\omega=1$.

We note that in the literature it has been common to discuss the validity or lack of validity of light-cone dominance in terms of phase oscillations in the Fourier integral.[3] Since by translation

$$\int d^4x\,e^{-iq\cdot x}\langle 0|T[A(x)B(0)]|p\rangle$$
$$=\int d^4x\,e^{-ik\cdot x}\langle 0|T[A(0)B(-x)]|p\rangle,$$

and $\exp(-ik\cdot x)\to\exp\{-i[x_+\nu-x_-k^2(2\nu)^{-1}]\}$ in the limit $\nu\to\infty$ (where $x_\pm=x_0\pm\vec{x}\cdot\hat{k}$), the phase is bounded if $x_+\lesssim\text{const}\,\nu^{-1}$ and $x_-\lesssim\text{const}\,\nu(k^2)^{-1}$. This only implies $x_+x_-\lesssim\text{const}(k^2)^{-1}$, and does not imply $x_+x_-\to 0$. Hence the phase oscillation argument applied to the right-hand side does not imply light-cone dominance in the A limit with $\omega=1$; whereas applied to the left-hand side it does. However, it is well known that for one-dimensional Fourier transforms the asymptotic behavior is controlled by those points for which the integrand or its derivatives develop singularities (the end points of Fourier integrals over finite intervals being included by using θ functions), and not necessarily by points for which the phase remains bounded, i.e., the neighborhood of the origin. Thus, the phase oscillation argument can be misleading. Since for the physical cases the singularities of the integrand are expected to be on the light cone, one might think that the light cone still controls the asymptotic behavior regardless of whether the phase remains bounded. However, because of the multidimensional nature of the Fourier integral at hand, leading singularities need not dominate. The above example $M(x\cdot p,x^2)$ provides a simple and explicit illustration of this point.

The question remains as to whether this phenomenon is only a mathematical possibility or whether it is physically relevant. Our next reason for considering the hypothesis of leading light-cone singularity dominance to be less compelling in vertex functions and the Regge limit of scattering amplitudes is based on considerations of the physical pole structure. Let us compare the vertex function V defined by

$$V(q\cdot p,q^2)$$
$$=-\int d^4x\,e^{-iq\cdot x}(k^2-1)\langle 0|T[A(x)\varphi(0)]|p\rangle \tag{4}$$

with the alternative form

$$\bar{V}(q\cdot p,q^2)=\int d^4x\,e^{-iq\cdot x}\langle 0|T^*[A(x)S(0)]|p\rangle, \tag{5}$$

where $S(x)=(\Box+1)\varphi(x)$; with the T^* product suitably defined to take into account equal-time terms, $\bar{V}=V$. Since $\dim S=\dim\varphi+2$, whereas the quantum numbers of S are identical to those of φ, if

$$A(x)\varphi(0)\underset{x^2\to 0}{\sim}(x^2-i\epsilon x_0)^{-r}\sum_n x^{\alpha_1}\cdots x^{\alpha_n}\mathcal{O}_{\alpha_1\cdots\alpha_n}(0), \tag{6}$$

then, according to the present rules,[4]

$$A(x)S(0)\underset{x^2\to 0}{\sim}(x^2-i\epsilon x_0)^{-r-1}\sum_n x^{\alpha_1}\cdots x^{\alpha_n}P_{\alpha_1\cdots\alpha_n}(0), \tag{7}$$

where $P_{\alpha_1\cdots\alpha_n}$ has the same dimension as $\mathcal{O}_{\alpha_1\cdots\alpha_n}$. If we write, in view of Eqs. (6) and (7),

$$\langle 0|T[A(x)\varphi(0)]|p\rangle=(x^2-i\epsilon)^{-r}\sum_{n=0}^{r-1}f^{(n)}(x\cdot p,0)(x^2)^n\frac{1}{n!}+F(x\cdot p,x^2),$$

$$\langle 0|T[A(x)S(0)]|p\rangle=(x^2-i\epsilon)^{-r-1}\sum_{n=0}^{r}g^{(n)}(x\cdot p,0)\frac{(x^2)^n}{n!}+G(x\cdot p,x^2),$$

where F and G do not have algebraic singularities on the light cone,[5] then one has in the A limit[6]

$$V(q\cdot p,q^2)\propto(k^2-1)\left\{\sum_{n=0}^{r-1}\frac{(i)^{r-n}(2\nu)^{r-2-n}}{(r-n-1)!\,n!}\int_0^\infty d\lambda\,\lambda^{-r+1+n}f^{(n)}(\lambda,0)\right.$$
$$\left.\times\exp\left[i\left(1-\frac{k^2}{2\nu}+O(\nu^{-2})\right)\lambda\right]+\tilde{F}(q\cdot p,q^2)+\cdots\right\}, \tag{8}$$

and

$$\bar{V}(q\cdot p,q^2)\propto\sum_{n=0}^{r}\frac{i^{r+1-n}(2\nu)^{r-1-n}}{(r-n)!\,n!}\int_0^\infty d\lambda\,\lambda^{-r+n}g^{(n)}(\lambda,0)\exp\left[i\left(1-\frac{k^2}{2\nu}+O(\nu^{-2})\right)\lambda\right]+\tilde{G}(q\cdot p,q^2)+\cdots. \qquad (9)$$

Thus for fixed k^2, if only the first terms in Eqs. (8) and (9) dominate, there is an apparent difficulty.[7] This difficulty can be removed in several ways. If the leading term in the field expansion describes correctly the asymptotic behavior, then necessarily

$$\int_0^\infty \lambda^{-r}g^{(0)}(\lambda,0)e^{i\lambda}d\lambda=0, \qquad (10)$$

in which case light-cone dominance in the source expansion must be understood in the sense that the next singular term has to be kept; another possibility is that some of the integrals $\int_0^\infty d\lambda \times\lambda^{-r+1+n}f^{(n)}(\lambda,0)e^{i\omega\lambda}$ diverge[8] at $\omega=1$ for large λ to give additional powers of ν, and/or the $\tilde{F}$ term is important. Whatever the case, the $\tilde{F}$ term must be important if the ultimately dominant term is not to have a zero at $k^2=1$. This is because, on account of the functional form of the exponent $\exp\{i\lambda[1-k^2/\nu+O(\nu^{-2})]\}$, each individual integral over λ can at best lead to a pole at $k^2=0$, not at $k^2=1$ (for general values of ν).

Thus the main possibilities can be summarized as follows:

(i) The rules of leading light-cone singularity dominance hold for the field expansion, and $V(q\cdot p,q^2)\to(k^2-1)\nu^{r-2}+O(\nu^{r-3})$. Then, the power of ν in the asymptotic behavior naively expected from the leading light-cone singularity of the source expansion is incorrect, and a sum rule of the form of Eq. (10) must operate (relying instead on cancelations from the $\tilde{G}$ term would be further from the spirit of light-cone dominance). A factor k^2-1 must then develop to multiply the next power of ν in the source expansion. Thus, the distinguishing feature of this possibility is that there is an abrupt change in the asymptotic behavior as k^2 moves from on shell to just slightly off shell. This is the behavior proposed in Ref. 1.

(ii) The asymptotic behavior does not change abruptly when k^2 goes off shell. This implies that at least in the field expansion the much less singular term $F(x\cdot p,x^2)$ must contribute as importantly as the leading singular term, and the light-cone singularities fail to dominate the asymptotic behavior. Furthermore, it follows from the discussion after Eq. (9) that without further specifications, not only the coefficient of the leading term but also the power of ν of the leading term is not reliably obtained from considering the light-cone singularity alone.

Note that in the A limit with $\omega\neq1$, (8) and (9) have the same leading power of ν, and the question of apparently different powers of ν in the field and source expansions does not arise. This again explains why we emphasize that the A limit with $\omega\neq1$ and with $\omega\to1$ are on a different footing.

The above considerations also apply to more general situations whenever a light-cone analysis based on a source-product expansion requires different handling from that based on a source expansion.[9] In particular, they are pertinent for the applications to a fixed-t scattering amplitude T for $A(k)+C(p)\to B(k')+D(p')$. With $\nu=2k\cdot p$, $t=(p-p')^2$, $\delta=2k\cdot(p-p')=(k'^2-k^2-t)$, in the A limit ($\nu\to\infty$; k^2/ν, δ/ν, and t fixed), the light-cone singular terms lead to

$$T\underset{A}{\to}(k^2-m_A^2)\sum_i\nu^{r-2-i}\int_0^\infty d\lambda\,\exp\left(i\frac{\lambda k^2}{2\nu}\right)f_i\left(\lambda,\frac{\lambda\delta}{\nu},t\right), \qquad (11)$$

$$\bar{T}\underset{A}{\to}\sum_i\nu^{r-1-i}\int_0^\infty d\lambda\,\exp\left(i\frac{\lambda k^2}{2\nu}\right)g_i\left(\lambda,\frac{\lambda\delta}{\nu},t\right), \qquad (12)$$

where T is obtained from the field expansion and $\bar{T}$ from the source expansion. Brandt and Orzalesi have proposed[10] that for the particular point $\delta=0$, the integrals $\int_0^\infty d\lambda\,\lambda^j g_i(\lambda,0,m_A^2-m_B^2)$ converge up to the first nonvanishing moment, so that the on-shell Regge limit $\delta=0$ ($t=m_A^2-m_B^2$) can be obtained by taking the limit $k^2\to m_A^2$ inside the integrals. From this an interesting quantization rule for Regge intercepts emerges.

We note that the terms not explicitly included in (11) must of course contribute when one sets $\delta=0$ and then lets $k^2\to m_A^2$, in order to obtain the Regge limit of T; thus the procedure of using light-cone dominance together with a continuation to the Regge limit cannot be valid for the field expansion, even for $\delta=0$. Furthermore, if we set $k^2=m_A^2+\epsilon$, $k'^2=m_B^2+\epsilon$, and $t=m_A^2-m_B^2$, then still $\delta=0$, so that the above assumptions would imply that $\overline{T}(\nu, m_A^2+\epsilon, 0, m_A^2-m_B^2)$ has the same asymptotic behavior as $\overline{T}(\nu, m_A^2, 0, m_A^2-m_B^2)$. For $t=m_A^2-m_B^2$, one is thereby led to expect no abrupt change in asymptotic behavior, when both A and B are off shell by the same amount.

Finally, we remark that the question of the extent to which an abrupt change in the asymptotic behavior on and off shell is directly observable and compatible with general theoretical considerations seems to be a very interesting one, and deserves further study on its own merit.[11]

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[1]R. A. Brandt and G. Preparata, Phys. Rev. Lett. 25, 1530 (1970), and CERN Report No. TH.1208, 1970 (to be published).

[2]Another possibility for the second term is $\exp(-ip\cdot x)\int \rho(a)\Delta_F(x^2,a)\,da$, with the first two moments of ρ vanishing. There are also functions entire in x^2, whose Fourier transforms approach constants as $\nu\to\infty$ with $\omega\to 1$. Although in these examples the asymptotic behavior in ν is not smooth as $\omega\to 1$, one can find examples such that although $\widetilde{M}$ has a different asymptotic behavior for $\omega\neq 1$ and $\omega=1$, $(k^2-p^2)\widetilde{M}$ is smooth in ω as $\omega\to 1$.

[3]R. A. Brandt, Phys. Rev. D 1, 2808 (1970); Y. Frishman, to be published.

[4]See, for example, R. A. Brandt and C. A. Orzalesi, to be published.

[5]For simplicity we are considering r to be an integer, as it is in perturbation theory. For half-integral values of r the following discussion can be appropriately modified.

[6]The finite number of terms not explicitly written out in Eq. (8) and Eq. (9) are of the form

$$(\nu)^{r-2-n}\int_{-\infty}^{0} d\lambda\, \lambda^{-r+1+n} f^{(n)}(\lambda,0)e^{-i\lambda[2\nu+O(1)]},$$

so that they do not change the argument that follows.

[7]One might think at first that it is the equal-time term arising from bringing the d'Alembertian past the time-ordering operator that accounts for the difference, but this is not the case since one can compare the matrix elements without the time ordering and the same apparent discrepancy arises there also.

[8]Without additional constraints on $f^{(n)}$ and $g^{(n)}$, some integrals also appear to diverge at $\lambda=0$. This is spurious, because one really should not simply replace the factor $(x^2-i\epsilon x_0)^{-r}$ by $(x^2-i\epsilon)^{-r}$ in going from the product expansion (6) or (7) to time-ordered product expansions; $(x^2-i\epsilon)^{-r}$ is not a well-defined generalized function for integers $r\geq 2$, whereas the matrix element of $A(x)\varphi(0)$ is. The symbol $(x^2-i\epsilon)^{-r}$ in the text is meant to represent a regularized distribution (with subtraction of a quantity concentrated at the origin). The integrals over λ are then defined at $\lambda=0$.

[9]Our comments are also relevant to the recent applications to exclusive processes by Y. Frishman, V. Rittenberg, H. R. Rubinstein, and S. Yankielowicz, Phys. Rev. Lett. 26, 798 (1971). In particular, it follows from our discussion that even if canonical dimensionality is correct, the powers d_i of ν in their Eq. (1) may be different from what is naively expected because of sum rules. Thus, if experimentally the d_i turn out to be different from zero, this need not be taken as evidence against canonical dimensionality.

[10]See Ref. 3. [The index i in Eq. (13) and Eq. (14) takes on half integers in that reference, but this point is not important for the present discussion.] See also R. Brandt, New York University Report No. 4171, 1971 (to be published), for further arguments on the special significance of the point $\delta/\nu=0$.

[11]As an example, consider the case of the pion electromagnetic vertex function $\Gamma_\mu=(k+p)_\mu A+(k-p)_\mu B$, with $p^2=1$. The Ward identity implies $q^\mu\Gamma_\mu=(k^2-1)A+q^2B=k^2-1$. Thus $B=-(k^2-1)(A-1)/q^2$ has the form of a leading term coming from the field expansion. However, B vanishes identically on shell and will not contribute to any process in which the virtual photon emitted by the pion is absorbed by a system making an on-shell transition. The interesting question is whether such a factor k^2-1 can appear in the leading asymptotic behavior of physically more accessible amplitudes M [e.g., $M\sim C_0(k^2-1)/q^2+C_1/q^4$].

[12]C. Orzalesi and P. Raskin, unpublished.