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Eikonal Expansion*†

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An eikonal expansion is considered which yields an impact-parameter representation for the scattering T matrix. The dependence on momentum transfer is summed to all orders by means of a conjecture to obtain a standard impact-parameter representation. As a result, phase-corrected eikonal formulas are obtained which are shown to improve on the Glauber approximation systematically at large scattering angles, in contrast to the Abarbanel-Itzykson approximation. Regge behavior is found.

The best known development of eikonal scattering theory is that of Glauber.[1] Glauber's well-known formula takes the form of a two-dimensional Fourier transform of the impact-parameter representation of the T matrix:

$$T = vi\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\, T(\vec{b}), \tag{1}$$

$$T(\vec{b}) \approx T^{(0)}(\vec{b}) = \exp[i\chi_0(b)] - 1. \tag{2}$$

The eikonal phase $\chi_0(\vec{b})$ is an integral over the interaction in the z direction which is chosen to be parallel to the average momentum, $\vec{k} = \frac{1}{2}(\vec{k}_i + \vec{k}_f)$:

$$\chi_0(\vec{b}) = -\int_{-\infty}^{\infty}(dz/v)\,V(\vec{r}). \tag{3}$$

In a very interesting approach reported in this Journal, Abarbanel and Itzykson[2] have suggested, for nearly forward scattering, a simple eikonal expansion about the average momentum. Their approximation is, to lowest order,

$$T_{\rm AI}(b) = \cos\tfrac{1}{2}\theta \exp[i\chi_0(b)/\cos\tfrac{1}{2}\theta] - 1, \tag{4}$$

where the only difference involves replacing the velocity v by $v\cos\frac{1}{2}\theta$. The result is, however, less fortunate than (2) at large angles.

This Letter summarizes an expansion which is related to that of Abarbanel and Itzykson, but is based more closely on the intuitive assumptions of Glauber. The eikonal expansion of the T matrix is a perturbation series in powers of the operator N which represents the defect of the eikonal propagator,

$$g^{-1}=\vec{v}\cdot(\vec{k}-\vec{P})-V+i\eta,\quad \vec{v}=(K/M)\hat{z}, \tag{5}$$

from the exact propagator

$$G^{-1}=K^2/2M-P^2/2M-V+i\eta. \tag{6}$$

This defect is written

$$N=g^{-1}-G^{-1}=\lambda(g^{-1}+V)+N_\tau, \tag{7}$$

where

$$N_\tau=(\vec{P}-\vec{k}_f)\cdot(\vec{P}-\vec{k}_i)/2M \text{ and } \lambda=1-\cos\tfrac{1}{2}\theta. \tag{8}$$

The formal expansion for T is

$$T=(V+VgV)+VgNgV+VgNgNgV+VgNgNgNgV+\cdots, \tag{9}$$

where g has been chosen so that the matrix element of the zero-order term, $T^{(0)}=\langle\vec{k}_f|V+VgV|\vec{k}_i\rangle$, is Glauber's formula. This eikonal series is perhaps less intuitive than the symmetrized expansion of Sugar and Blankenbecler[3] but has the clear advantage of automatically yielding an impact-parameter representation which takes the form

$$T(\vec{b})=(1-\lambda)T^{(0)}(\vec{b})+\mathcal{T}^{(1)}(\vec{b})+\sum_{n=2}^{\infty}\sum_{m=2}^{n}\lambda^{n-m}\binom{n-2}{m-2}\mathcal{T}^{(m)}(\vec{b}), \tag{10}$$

where $\binom{n-2}{m-2}$ is the binomial coefficient. Four parameters are important in the expansion: q, the moment transfer ($q=2K\sin\frac{1}{2}\theta$); KR, the number of projectile wavelengths inside the range R of the interaction; Ka, the change in the interaction in one projectile wavelength ($|\nabla V|\approx|V|/a$); and $\epsilon=V_0/2E$, the ratio of interaction strength to twice the center-of-mass energy.

When the interaction is a spherically symmetric potential, the first three eikonal corrections are reducible to tractable forms which are summarized by the following equations:

$$\mathcal{T}^{(1)}(b)=\exp[i\chi_0(b)][i\lambda\chi_0(b)+i\tau_1(b)]; \tag{11}$$

$$\mathcal{T}^{(2)}(b)=\exp[i\chi_0(b)]\{(q^2/8K^2)[1-i\chi_0(b)]+[i\lambda\chi_0(b)+i\tau_1(b)]^2/2!+2i\lambda\tau_1(b)+i\tau_2(b)-\omega_2(b)\}; \tag{12}$$

$$\begin{aligned}\mathcal{T}^{(3)}(b)=\exp[i\chi_0(b)]\{&-(q^2/8K^2)[2+i\chi_0(b)][i\tau_1(b)]+(\lambda q^2/8K^2)\chi_0^2(b)+[i\lambda\chi_0(b)+i\tau_1(b)]^3/3!+i\lambda^2\tau_1(b)\\&+3i\lambda\tau_2(b)-2\lambda\omega_2(b)+[i\lambda\chi_0(b)+i\tau_1(b)][i\lambda\tau_1(b)+i\tau_2(b)-\omega_2(b)]+i\tau_3(b)+i\varphi_3(b)-\omega_3(b)\}.\end{aligned} \tag{13}$$

The objects appearing in (11)–(13) can be calculated from the following formulas in which we write the spherically symmetric potential as $V(r)=V_0U(r)$ and employ the transverse derivative operations $\beta_n\equiv b^n\partial^n/\partial b^n$:

$$\chi_0(b)=-2K\epsilon\int_0^\infty dz\,U(r), \tag{14}$$

$$\tau_1(b)=-K\epsilon^2(1+\beta_1)\int_0^\infty dz\,U^2(r), \tag{15}$$

$$\tau_2(b)=-K\epsilon^3(1+\tfrac{5}{3}\beta_1+\tfrac{1}{3}\beta_2)\int_0^\infty dz\,U^3(r)-b[\chi_0'(b)]^2/24K^2, \tag{16}$$

$$\tau_3(b)=-K\epsilon^4(\tfrac{5}{4}+\tfrac{11}{4}\beta_1+\beta_2+\tfrac{1}{12}\beta_3)\int_0^\infty dz\,U^4(r)-b\tau_1'(b)[\chi_0'(b)]^2/8K^2, \tag{17}$$

$$\varphi_3(b)=-K\epsilon^2(1+\tfrac{5}{3}\beta_1+\tfrac{1}{3}\beta_2)\int_0^\infty dz[(2K)^{-1}\partial U(r)/\partial r]^2, \tag{18}$$

$$\omega_2(b)=b\chi_0'(b)\nabla^2\chi_0(b)/8K^2, \tag{19}$$

$$\omega_3(b)=[b\chi_0'(b)\nabla^2\tau_1(b)+b\tau_1'(b)\nabla^2\chi_0(b)]/8K^2. \tag{20}$$

These formulas yield analytic results for Coulomb, Yukawa, and Gaussian potentials, and we find the remarkable fact that all of the eikonal phases except $\chi_0(b)$ vanish for a Coulomb potential.

The higher-order eikonal corrections are prohibitively complicated with the exception of a simple term, appearing in all orders, of the form

$$\exp[i\chi_0(b)][i\lambda\chi_0(b)+i\tau_1(b)]^n/n!. \tag{21}$$

The correction terms starting in $T^{(2)}$ which involve $q^2/8K^2$ arise from integrating by parts terms involving $-\nabla^2/8K^2$. These terms are conjectured to cancel the class of terms involving

$$\lambda = 1-(1-q^2/4K^2)$$
$$= q^2/8K^2 + q^4/128K^4 + q^6/1024K^6 + \cdots \qquad (22)$$

in a power-by-power way. The beginning of this behavior can be seen in Eqs. (11)–(13); it must continue if the eikonal expansion is to retain the virtue of the Glauber term $T^{(0)}$ by yielding an exact Coulomb-scattering result. If we assume that the conjecture is correct, the results given above simplify at all momentum transfers to the form they take at $q=0$. In this limit, we find an ordered set of corrections to the Glauber approximation to be

$$T^{(\mathrm{I})}(\vec{b}) = \exp\{i[\chi_0(b)+\tau_1(b)]\}-1, \qquad (23)$$

$$T^{(\mathrm{II})}(\vec{b}) = \exp\{i[\chi_0(b)+\tau_1(b)+\tau_2(b)]\} \times \exp[-\omega_2(b)]-1, \qquad (24)$$

$$T^{(\mathrm{III})}(\vec{b}) = \exp\{i[\chi_0(b)+\tau_1(b)+\tau_2(b)+\tau_3(b)+\varphi_3(b)]\} \times \exp\{-[\omega_2(b)+\omega_3(b)]\}-1, \qquad (25)$$

where exponentiation of $\tau_1(b)$ is justified to all orders. Exponentiation of $\tau_2(b)$ and $\tau_3(b)$ is justified through third order in the expansion and is motivated by the resulting correspondence with the expansion of the WKB phase in powers of $\epsilon U(r)$. The exponentiation of the real phases $\omega_2(b)$ and $\omega_3(b)$ is justified through third order in the expansion and is motivated by the unitarity requirement $|S(b)|\neq 1$ $[S(b)=T(b)+1]$ for the impact-parameter representation (1).

Elimination of the explicit q^2 dependence from $T(b)$ puts the above phase-corrected eikonal amplitudes in direct correspondence with the standard impact-parameter representation[4] associated with the inverse Fourier transform of $T(q^2)$. This is, in effect, the underlying symmetry which we have conjectured in summing the q dependence in the eikonal series (10).

Figure 1 shows the eikonal results for scattering by a Yukawa potential from (2) and (23)–(25) in comparison with (4), the Born approximation, and the partial-wave result. The parameters used in the calculation correspond to a collision of a slow proton (125 eV) with a hydrogen atom, assuming an interaction potential $V_0e^{-\mu r}/\mu r$ with strength $V_0 = 2$ Ry $= 54.4$ eV and $\mu = 2a_0^{-1}$. The eikonal amplitudes $T^{(\mathrm{I})}$, $T^{(\mathrm{II})}$, and $T^{(\mathrm{III})}$ are seen to converge to the magnitude and phase of the partial-wave scattering amplitude.

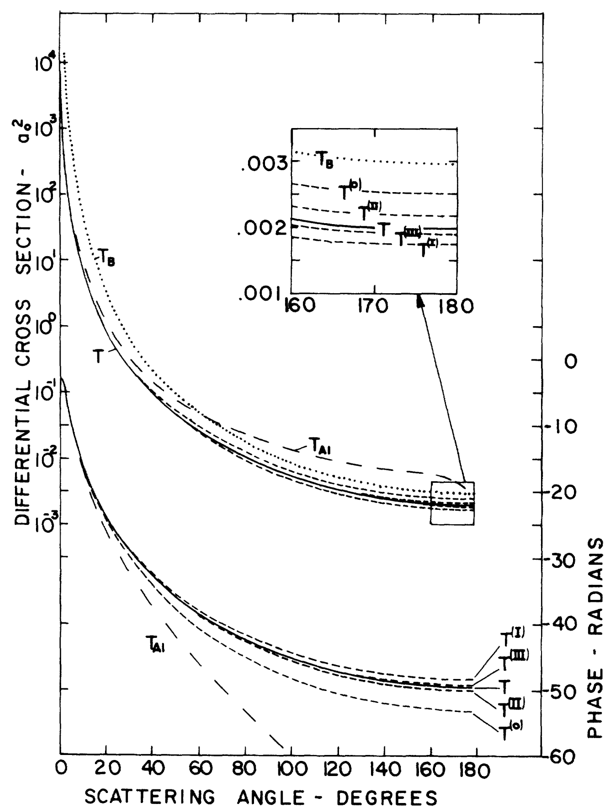

FIG. 1. Differential cross section and phase of scattering amplitude for Yukawa potential: T = partial wave, T_B = Born, T_{AI} = Abarbanel-Itzykson, $T^{(0)}$ = Glauber; $T^{(\mathrm{I})}$, $T^{(\mathrm{II})}$, and $T^{(\mathrm{III})}$ are the first, second, and third phase-correct eikonal, respectively.

Finally, we note that the dominant ln$b$ singularity in the Yukawa phases as $b\to 0$ gives rise to explicit Regge behavior for large q^2: $T^{(\mathrm{III})}(q^2) \propto q^{2\alpha}$, where

$$\alpha = [-1+\tfrac{1}{4}\epsilon^2 - \tfrac{5}{4}\epsilon^3] + i(K/\mu) \times[-\epsilon+\epsilon^2-2\epsilon^3+\tfrac{31}{6}\epsilon^4-\tfrac{1}{12}\epsilon^2(\mu/K)^2]. \qquad (26)$$

An explicit demonstration of unitarity for the phase-corrected eikonal amplitude (25) can be made using this result.

Numerical comparisons for potential scattering have also been made for Gaussian and Fermi potentials for which the phase-corrected eikonal formulas converge less uniformly to the exact results but do give a consistently improving result if the momentum transfer is not too large.

The eikonal expansion for potential scattering serves as a guide for the more interesting case of scattering by compound systems. The phase-corrected eikonal formula $T^{(\mathrm{I})}$ of (25) can be readily extended to such a case but a more complicated result is found.

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$s \leftrightarrow u$ Symmetry of s-Channel Helicity Amplitudes

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The $s \leftrightarrow u$ symmetry of s-channel helicity amplitudes is investigated *without the exchange mechanism being specified.* Conclusions are drawn concerning the possibility of phase assignment to scattering amplitudes at high energy. Difficulties are found in absorption models with $J_{\Delta\lambda}s^{\alpha}$ behavior.

Recently a number of models assigning a phase to s-channel helicity amplitudes at high energy have been proposed[1] *without the exact nature of the t-channel exchange mechanism* (such as fixed or moving poles, cuts, etc.) *being specified.* As is well known, such an assignment is indeed possible if an amplitude has definite $\nu \to -\nu$ symmetry $[\nu = \frac{1}{2}(s-u)]$ and ν^{α} behavior at high energy.[2] In this paper we use only C, T, P, and SU(2) [SU(3)] to find s-channel helicity amplitudes obeying the above mentioned symmetry.

Let us consider the s-channel reaction $ab \to cd$. Since $\nu \to -\nu$ correlates this reaction with $a\bar{d} \to c\bar{b}$ (u channel), it corresponds to the interchange of particles $\bar{d}$ and b in the t-channel reaction $\bar{d}b \to c\bar{a}$. Thus in order to obtain symmetry under $\nu \to -\nu$ for the s-channel helicity amplitudes, hereafter denoted by G, we shall first look for symmetry under $\bar{d} \leftrightarrow b$ in the t-channel helicity amplitudes, denoted by F.

We find that such symmetry exists for the t-channel helicity amplitudes in the following two cases: (a) Particles $\bar{d}$ and b belong to the same multiplet M of SU(2) (e.g., $p\pi^+ \to \Delta^{++}\pi^0$, $b=\pi^+$, $d=\pi^0$) or SU(3) (e.g., $pK^- \to \Lambda^0\pi^0$, $b=K^-$, $d=\pi^0$); (b) particles $\bar{d}$ and b belong to multiplets of SU(2) or SU(3) connected by change conjugation C, and particles c and $\bar{a}$ remain in their multiplets under C (e.g., $\pi^-p \to A_2^0 n$, $\pi^-p \to K^{*+}\Sigma^-$). For both cases the following symmetry holds for the t-channel helicity amplitudes:

$$F_{\lambda_c \lambda_a \lambda_d \lambda_b}{}^{i}(t,\nu) = (-1)^{\eta_i} F_{\lambda_c \lambda_a \lambda_b \lambda_d}{}^{i}(t,-\nu), \quad (1)$$

where i labels t-channel matrix elements reduced with respect to the internal symmetry [SU(2) or SU(3)], $\{\lambda\}$ denote the helicities of the particles in the t-channel, and $(-1)^{\eta_i} = \pm 1$ is a phase independent of helicities.

For case (a), Eq. (1) follows immediately from the statistics[3] and from the well-defined symmetry η_i of the irreducible representation $M \otimes M$ labeled by i. For case (b) one uses G and C invariance to prove Eq. (1). Consider for example, the t-channel reaction $\pi^+\rho^0 \to \bar{n}p$ (pure $I=1$). Using G parity and fermion anticommutation relations one easily obtains Eq. (1) with $\eta_i = +1$. A similar proof holds for all other class-(b) reactions in the SU(2) case. For b and $\bar{d}$ belonging to SU(3) octets one proves Eq. (1) [with i denoting the reduced amplitudes $\underline{1}$, $\underline{8}_{ss}$, $\underline{8}_{sa}$, $\underline{8}_{as}$, $\underline{8}_{aa}$, $\frac{1}{2}(\underline{10} \pm \underline{10}^*)$, or $\underline{27}$] by applying statistics and G parity to states of $I=0$ in $\underline{8} \otimes \underline{8}$.

Whenever Eq. (1) holds in the physical region of the t channel, we assume its validity for the whole analyticity domain of F in ν and t.

Let us now transform Eq. (1) from the t-channel helicity amplitudes F to the s-channel helicity amplitudes G, using the crossing relations