Numerical Simulation of Plasma Diffusion Across a Magnetic Field in Two Dimensions*

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Plasma computer simulation in two dimensions shows that there are three different regions for plasma diffusion across a magnetic field. These regions may be called collisional or classical, intermediate, and Bohm diffusions. In the intermediate region, which lies between the collisional and Bohm regions, the diffusion coefficient is essentially independent of the magnetic field. We believe that diffusion in the intermediate and the Bohm regions is caused by thermally excited convective motions.

It is well known that the classical collisional diffusion of plasma (or of test particles) across a magnetic field is proportional to $1/B^2$, where B is the strength of the magnetic field. The validity of this law is somewhat questionable for a two-dimensional plasma (a plasma of charged rods with B parallel to the rods) since particles can repeatedly collide with each other and the assumption of random independent collisions which is used to derive the $1/B^2$ law is called into question. In addition, Taylor and McNamara[1] have recently shown that diffusion in a two-dimensional plasma whose particles move only in accordance with the $c\vec{E}\times\vec{B}/B^2$ guiding center drifts should be of the Bohm type, i.e., the diffusion coefficient is proportional to $1/B$.

Here we should like to report on some numerical simulation and some theoretical analysis of the diffusion of test particles in a two-dimensional plasma. Both the simulation and the analysis include the full dynamics of particles. We find that there appear to be three regions. The first is a purely classical regime with the diffusion going as $1/B^2$. This situation occurs if the particles diffuse a distance equal to one Debye length in executing one gyro-orbit. It is clear that when this situation occurs, a particle encounters different particles on each orbit and the classical theory should hold. The second regime is one in which the diffusion is roughly independent of B. We believe that the diffusion here is due to random motion arising from thermally excited vortices. We shall present a rough theory for this. Finally, for very large B the diffusion appears to take on a $1/B$ dependence in accordance with the results of Taylor and McNamara.[1] This region also probably involves eddy diffusion.

In a plasma, these three diffusion processes take place simultaneously, and they can be classified in the following way. The criterion that the diffusion is dominantly due to collisions is given by

$$d \equiv \alpha(\lambda_D/\rho_e)(n\lambda_D^2)^{1/3} \lesssim 1, \tag{1}$$

where λ_D, ρ_e, and n are the Debye length, electron gyroradius, and the number density, respectively, and α is a numerical coefficient of order unity. This criterion is obtained as follows. If a particle diffuses a distance greater then a Debye length in executing a gyration, then it will encounter different particles on each gyration. All encounters can then be treated as random, independent events, and the classical $1/B^2$ diffusion should be obtained. The particle diffusion in velocity is given by

$$\Delta v^2 = v_t^2 t/\tau_c,$$

where v_t is the thermal velocity of the particle under consideration and τ_c is its collision time. Since for the cases we will consider the electron and ion masses will not differ by a large factor, we shall not distinguish between them in this argument. The collision time is given by[2]

$$\omega_p \tau_c = 16 n\lambda_D^2.$$

The finite size of the particles used (the half-width of the particle size is equal to the grid spacing) will increase τ_c by perhaps 50%. The distance a particle is deflected from its noninteracting orbit is roughly given by

$$(\Delta r)^2 = v_t^2 t^3/\tau_c.$$

If we set $t = 2\pi\omega_c^{-1}$ and $\Delta r = \lambda_D = v_t/\omega_p$, then the relation (1) is obtained. On the other hand, the intermediate region starts at $d \gtrsim 1$ and is followed by the Bohm region when $\rho_i/\lambda_D \lesssim 1$ is satisfied.

These conditions will be given later.

In the following we show results of numerical simulation and comparison with theory. The computer code used was a two-dimensional, electrostatic, dipole expansion code[3] with a uniform external magnetic field. Since the diffusion is due to fluctuations, we measured the time-averaged fluctuation spectrum in k space and compared it with the equilibrium theory; excellent agreement was observed for all calculations, confirming that the plasma was indeed close to thermal equilibrium.

Figure 1 shows results from four experiments and their comparison with theoretical predictions. In these experiments we used $\lambda_D = 5$, $\rho_e = 20$, and the mass ratio $m_i/m_e = 1.25$ on a 64×64 spatial grid. The number of particles varied as $n\lambda_D^2 = 6.25$, 25, 100, and 400, and correspondingly d changes as 0.46, 0.73, 1.16, and 1.84. (It is assumed that $\alpha = 1$.) Therefore, the first two experiments are expected to be in the collisional region, the third experiment is in a transition region, and the last experiment is in the intermediate region. The experiments indeed agree well with the above criterion. The first two follow closely the collisional theory[4] which gives

$$\frac{D_e}{\omega_{pe}} = \frac{1}{8}\left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_{pe}^2}{\omega_{ce}^2} \frac{1}{n}\left[1 + \left(\frac{2m_i}{m_i + m_e}\right)^{1/2}\right],$$

$$\frac{D_i}{D_e} = \left(\frac{m_i}{m_e}\right)^{1/2}\left[1 + \left(\frac{2m_e}{m_i + m_e}\right)^{1/2}\right] \times \left[1 + \left(\frac{2m_i}{m_i + m_e}\right)^{1/2}\right]^{-1}. \tag{2}$$

D_i and D_e are the ion and electron diffusion coefficients, respectively $[\langle(\Delta r)_{i,e}^2\rangle = D_{i,e}t]$. Since we are looking at test-particle diffusion, ion and electron diffusion need not be equal. For the third experiment the deviation becomes larger and for the fourth experiment both collisional and Bohm diffusion theory[1] show a large discrepancy with the experiment. These experiments confirm that in the collisional regime the diffusion is proportional to the collision frequency.

Figure 2 shows results of two experiments in the collisional region where the magnetic field was changed by a factor of 2. The ion diffusion follows closely the collision theory confirming that the diffusion follows $1/B^2$ in this region. However, the electron diffusion is somewhat larger than the theory predicts, probably because the electrons make the transition to the intermediate region sooner due to their smaller Larmor radii.

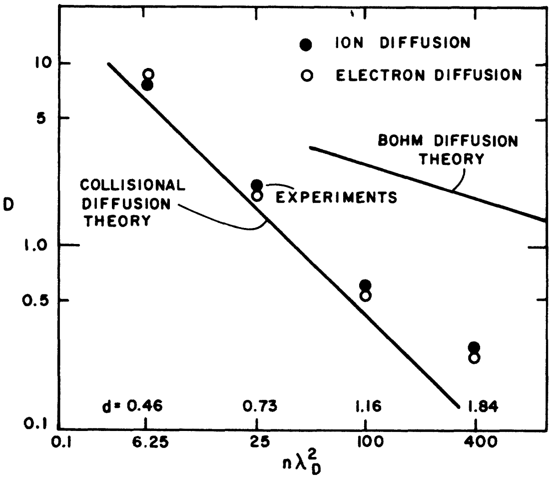

FIG. 1. Results of four experiments using different numbers of particles. When $d \lesssim 1$, diffusion follows close to classical theory. $\lambda_D = 5$, $\rho_e = 20$, $m_i/m_e = 1.25$, and 64×64 grid were used. Grid size was taken to be 1 throughout whole calculation. Unit of $D_{i,e}$ is in terms of grid size and electron plasma frequency. Length of calculation was $\omega_{pe}t = 400$.

Figure 3 shows the results of two different sets of experiments including both the intermediate and the Bohm regions. The Debye length was kept at 10 and the electron gyroradius was changed

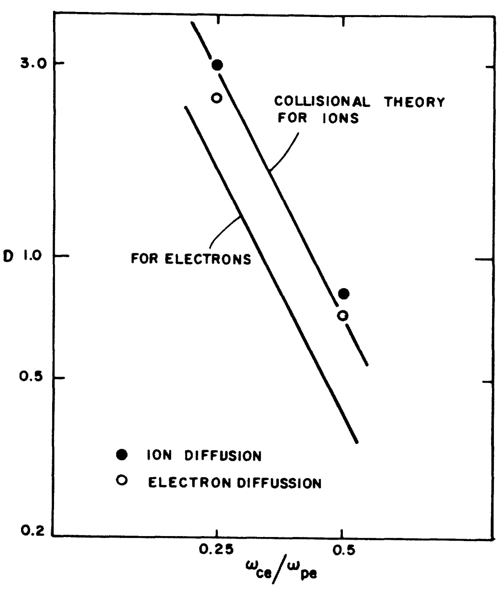

FIG. 2. Results of two experiments with magnetic field changing by factor of 2. Observed diffusion follows close to $1/B^2$ law in this classical diffusion region. $\lambda_D = \sqrt{2}$, $m_i/m_e = 9$, $n\lambda_D^2 = 2$, and 64×64 grid were used. Length of calculation was $\omega_{pe}t = 200$.

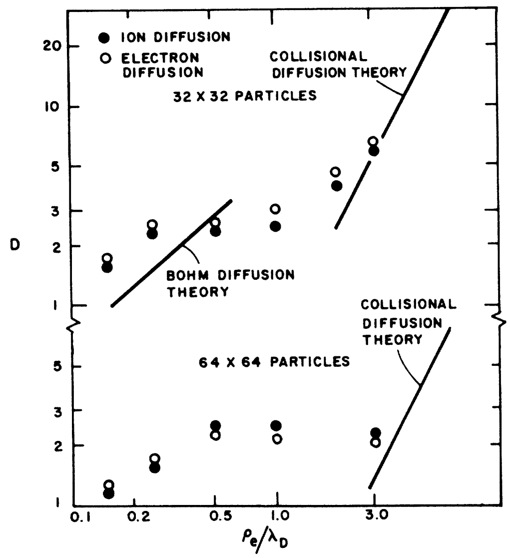

FIG. 3. Summary of two sets of experiments with changing magnetic field covering three regions. In first set of experiments 32×32 particles and in second 64×64 particles were used on 64×64 grids. $\lambda_D = 10$ and $m_i/m_e = 1.25$. Length of calculation was $\omega_{pe}t = 400$.

from 30 to 1.5. The number of particles used was 32×32 for each species in the first set of experiments, while 64×64 for the second set. In the intermediate region the diffusion coefficients are independent of the magnetic field for both cases. With the increase of the field the intermediate region is followed by the Bohm region where the diffusion goes as $1/B$.

The $\rho_e = 30$ case is close to the prediction of the collision theory for the first set, while the corresponding experiment for the second set is larger than the prediction of this theory. This is because of the larger value of d for the second set and it does not belong to the collisional region. As is given below, the diffusion in both the intermediate and the Bohm regions is close to the theoretical predictions. We believe that the diffusion in these regions is due to thermally excited vortices, plasma flow transverse to $\vec{k}$.

The following is a rough theory and a semiquantitative discussion of the effect. In thermal equilibrium all possible modes of the system are thermally excited. One set of modes is associated with vortex-type motion of the plasma. Associated with this transverse motion is a longitudinal field $\vec{E}$ of such a magnitude that $\vec{v}_T = c\vec{E}\times\vec{B}/B^2$. Let us Fourier analyze the transverse motion of the plasma as follows:

$$\vec{v}_T = \sum_{\vec{k}} \vec{v}_{Tk} \exp(i\vec{k}\cdot\vec{r}), \quad \vec{v}_{Tk}\cdot\vec{k} = 0.$$

Here we consider our system to be doubly periodic with periods L. The energy associated with this transverse motion is

$$W = L^2 \sum_{\vec{k}} \left(\frac{\rho|\vec{v}_{Tk}|^2}{2} + \frac{|\vec{E}_k|^2}{8\pi}\right) = \rho L^2\left(1+\frac{B^2}{4\pi\rho c^2}\right)\sum_k \frac{|\vec{v}_{Tk}|^2}{2}.$$

Now, according to equilibrium statistical mechanics, each of these transverse modes should have energy $k_BT/2$. Thus, we have

$$|\vec{v}_{Tk}|^2 = k_BT\left[\rho L^2\left(1+\frac{B^2}{4\pi\rho c^2}\right)\right]^{-1}. \tag{3}$$

In response to all these vortices a particle in the plasma is executing a random walk and the result will be a diffusion. The diffusion caused by any one mode is given by

$$|\Delta r_k|^2 = |\vec{v}_{Tk}|^2 \tau_{ck} t,$$

where τ_{ck} is the correlation time (or lifetime) for mode k. The total diffusion rate is

$$(\Delta r)^2 = \sum_k |\vec{v}_{Tk}|^2 \tau_{ck} t = Dt. \tag{4}$$

Now, each mode is continually being torn apart by the motion produced by all other modes. If this motion produces a diffusion then we may expect that the lifetime of mode k will be given by

$$\tau_{ck} = (k^2 D)^{-1}, \tag{5}$$

where D is the diffusion coefficient. Here perhaps we should use only the diffusion caused by modes with larger k's (shorter wavelengths) for computing the lifetime of mode k since longer wavelengths do not cause diffusion on the scale of $1/k$. However, this changes our result by only a relatively minor numerical factor.

Substituting the results from Eqs. (3) and (5) into Eq. (4) gives

$$\frac{k_BT}{\rho L^2(1+B^2/4\pi\rho c^2)}\sum_{\vec{k}}\frac{1}{k^2D} = D.$$

Converting the sum into an integral over k (the density of modes is $L^2kdk/2\pi$) gives

$$D = \frac{1}{(2\pi)^{1/2}}\left[\frac{k_BT}{n(m_i+m_e)[1+B^2/4\pi n(m_i+m_e)c^2]}\right]^{1/2} \times\left(\ln\frac{k_{\max}L}{2\pi}\right)^{1/2}. \tag{6}$$

For $B^2/4\pi\rho c^2$ small, this gives a diffusion coefficient which is independent of B. If this diffusion coefficient, in the classical regime, is smaller than that given by collisions, then collisions will dominate and mask this process. However, when that process becomes too small, this vortex diffusion will take over. The flat region in Fig. 3 is close to that predicted by Eq. (6).

This vortex diffusion is similar to the guiding-center diffusion discussed by Taylor and McNamara.[1] However, in this intermediate regime the electric field associated with the vortex motion is just proportional to B, and so B cancels out of the diffusion formula. One cannot use the expectation value of $\langle |E_k|^2\rangle^{1/2}$, as Taylor and McNamara did, for the following reason. While the expected electric field energy is $\frac{1}{2}k_BT$ per k, for $k\lambda_D \ll 1$ (independent of B) this energy is shared between the low-frequency vortex motion and high-frequency oscillations at the hybrid frequency and at the cyclotron harmonics. It is not proper to treat the motion of a particle as being a guiding-center drift in the high-frequency fields. This is permissible only for the low-frequency ($\omega = 0$) vortex-type motion.

When B becomes large, the electric field fluctuations are primarily associated with the vortex motion. The magnetic field is responsible for the high frequency of the hybrid fluctuations, and they give rise to little charge-density or electric-field fluctuations. In this regime the diffusion goes as $1/B$, which is Bohm-like. However, it is proportional to $\sqrt{T}$ rather than T. The diffusion also depends logarithmically on the size of the system because large vortices live a long time and can convect particles a long way. This dependence of the diffusion on T and B is the same as that found by Taylor and McNamara.[1]

Experimentally we have confirmed that these vortex modes play an essential role for the plasma diffusion in both the intermediate and the Bohm regions. When these modes were suppressed artifically, then the diffusion decreased drastically and tended to approach to the value predicted by the collision theory.

Finally, we should like to discuss the relation between two- and three-dimensional diffusions. While it is not clear how important vortex diffusion will be in three-dimensional plasma, it is clear that it can occur there, and the classical theory should be modified for large fields. Such motion has been observed in a number of experiments,[5-7] and in these it appears to play an important role in plasma transport. It appears that if the vortex motions are only thermally excited, they will not cause a serious plasma loss; however, if their excitation is much larger than the thermal level, they could play a very important role in plasma transport. A more complete theory with detailed experimental results will be published soon.

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