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Low-Temperature Behavior of the Quantum Cluster Coefficients*

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We show that the leading low-temperature behavior of the nth cluster coefficient b_n can be calculated without solving the n-body problem, and that there is a $\ln T$ term in the expansion. We illustrate these results particularly for the case of b_3.

Dashen, Ma, and Bernstein[1] have shown recently how to express the grand canonical potential in terms of the S matrix. They obtain expressions for the nth cluster coefficient b_n in terms of energy integrals of traces of the S matrix weighted by $e^{-\beta E}$. For low temperatures (large $\beta=1/kT$) these integrals are dominated by the low-energy behavior of the scattering amplitude. We have shown recently that this low-energy behavior can be obtained exactly for the n-body system without solving the n-body problem.[2] In this note we combine these two ideas to obtain explicitly the leading terms in the low-temperature expansion of the third cluster coefficient b_3 for identical bosons.[3] We show how higher terms can be computed, and that there are $\ln T$ terms among them. We also obtain the form of the leading low-temperature term for b_n in general.

We begin with the standard expression for the grand canonical potential in terms of the cluster coefficients b_n[4]:

$$\beta\Omega=-\ln \mathrm{Tr}e^{-\beta(H-\mu N)}=-\frac{V}{\lambda^3}\sum_{n=2}^{\infty}b_n e^{n\beta\mu}+\beta\Omega_0, \qquad (1)$$

where $\lambda=(2\pi\beta/m)^{1/2}$ and Ω_0 is the ideal-gas part of Ω. We use $\hbar=2m=1$. Dashen, Ma, and Bernstein have shown that

$$b_n-b_n^{(0)}=n^{3/2}\int dE\, e^{-\beta E}\frac{1}{2\pi}\frac{\partial}{\partial E}\,\mathrm{Im}[\mathrm{Tr}_n A\ln S(E)]_c, \qquad (2)$$

where $b_n^{(0)}$ is the nth cluster coefficient without interactions, Tr_n means trace in the n-body space, A instructs us to take that trace with respect to correctly symmetrized (or antisymmetrized) states, and the subscript c on the right stands for connected; $S(E)$ is the S matrix at energy E. Since the c is not on the S, we must include states connected by the symmetrization as well as by the scattering itself. It is a straightforward exercise in scattering theory to rewrite (2) as

$$b_n - b_n^{(0)} = -\frac{n^{3/2}}{\pi}\int dE\, e^{-\beta E}\frac{\partial}{\partial E}\mathrm{Re}\{\mathrm{Tr}_n A \arctan[\pi\delta(E-H_0)K(E)]\}_c, \tag{3}$$

where K is the K matrix, defined by

$$K = v + v\,\mathrm{P}(E-H_0)^{-1}K \tag{4}$$

where P stands for principal value. This transformation to the K matrix removes some, but not all, of the technical problems encountered by Dashen and Ma associated with singular terms.[5] For the two-body system of identical bosons at low energies, the K matrix reduces to $16\pi a$ (a is the two-body scattering length). In expanding $\arctan[\pi\delta(E-H_0)K(E)]$ in the two-body case ($\arctan x = x - \frac{1}{3}x^3 + \cdots$), each intermediate integration involves $d^3q\,\delta(E-2q^2)$ which gives a factor of $E^{1/2}$ on dimensional grounds. Hence we need only keep $\arctan x \sim x$ for small E. Applied to (3) this gives the familiar low-temperature answer

$$b_2 - b_2^{(0)} = -2a/\lambda + O(1/\lambda^3). \tag{5}$$

In the three-body case there are two types of connected contributions to the trace. There are those connected only by statistics, of which Fig. 1(a) is the simplest example, and there are those connected by the interactions. The leading low-temperature contribution of Fig. 1(a) to b_3 is $-\sqrt{2}a/\lambda$. Dashen, Ma, and Bernstein have shown that terms such as in Fig. 1(a) may be considered as statistics-generated corrections to b_2, and they are automatically included in b_2 if b_2 is redefined so that the trace is carried out with the states weighted by the usual Fermi or Bose statistical weight. Hence we shall concentrate on the contributions to b_3 coming from genuine three-particle scatterings, such as Fig. 1(b). Let us denote the sum of these contributions as $\bar{b}_3$.

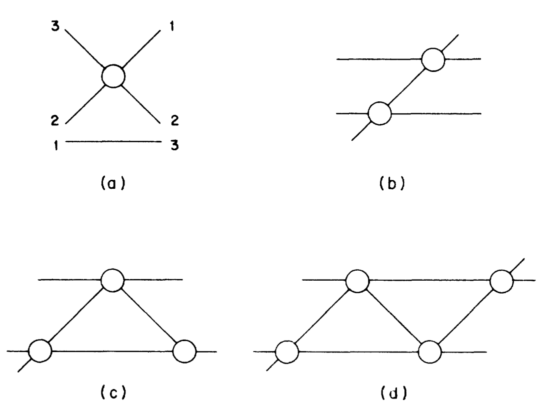

FIG. 1. Lowest-order multiple-scattering three-body diagram in the connected boson K matrix. (a) A term connected only by symmetrization. (b)–(d) Terms connected by scattering. These terms have different functional forms and weights as perturbation of the labels are assigned over them. In all cases the circles represent the off-shell two-body K matrix.

We have shown[2] recently that the connected three-body T matrix has the low-energy expansion

$$A/E + B/E^{1/2} + C\ln E + O(1), \tag{6}$$

where A, B, and C contain only two-body scattering lengths and kinematical factors. For the K matrix the arguments are even simpler, and the expansion is still valid. A goes with the low-energy expansion of Fig. 1(b), B with 1(c), and C with 1(d). A, B, and C are real and proportional to a^2, a^3, and a^4, respectively. As in the two-body case it is easy to see, essentially on dimensional grounds, that the leading small-E term in the arc tan comes from the linear term ($\arctan x \sim x$), and that the leading term is then the A term of (6). Using the explicit form of A gives the leading low-temperature contribution to $\bar{b}_3$,

$$\bar{b}_3 = 10a^2/\lambda^2 + O(1/\lambda^3) \tag{7}$$

This agrees with the result of Pais and Uhlenbeck[6] obtained by a much more complex method, and also with the λ dependence of the perturbation result of Larsen and Mascheroni.[7] It also is easily converted to give the hard-sphere result of Lee and Yang[8] and serves as a proof of the often-stated fact that the leading low-temperature terms of the hard-sphere result are valid in general with the scattering length replacing the hard-sphere radius.[9]

In evaluating the trace for (7) one encounters the singular terms of the form

$$\int f(x)\delta(x)\,\mathrm{P}(1/x)\,dx;$$

using the standard limiting procedure [e.g., $1/(x-i\epsilon) = \mathrm{P}(1/x) + i\pi\delta(x)$], this integral is easily shown to be $\frac{1}{2}[df/dx]_{x=0}$. The evaluation of higher terms is rather complex because of the many terms introduced by symmetrization, and we have not been able to obtain an answer in closed form although it is clear on dimensional grounds that the next term goes like $1/\lambda$.[3] To get this term one not only keeps the contribution of Fig.

1(c) to the linear term in the arc tan but one must also take the cubic term in the arc tan. There is a connected term of precisely the same geometry as Fig. 1(c) coming from the cube of the disconnected or two-body parts of the K matrix. Here, however, intermediate states get a $\delta(E-H_0)$ rather than a $P[1/(E-H_0)]$. It is important to take these two terms together in order to get a finite answer. All other terms are higher order.

Putting the C term of Eq. (6) [Fig. 1(d)] in the linearized arc tan makes a direct evaluation even more difficult, but it is again clear on dimensional grounds that there will be a term in $\bar{b}_3$ of the form $\lambda^{-4}\ln\lambda$. It has been claimed that b_n contains only positive and negative integral powers of λ.[10] This does not seem to be the case.

It is clear from this analysis that terms beyond $\lambda^{-4}\ln\lambda$ in $\bar{b}_3$ require a full solution of the three-body problem since they involve terms of $O(1)$ in E in Eq. (6).

Turning to the n-body cluster coefficient b_n, there will again be terms connected by statistics, which presumably can be summed into statistical corrections to lower-order terms and terms connected by scattering. It is easy to see that the leading low-energy contribution to the n-body K matrix goes as A/E^{n-2} with A proportional to a^{n-1}. In Eq. (3) again the linear term in the arc tan will dominate. Thus one finally gets as the leading low-temperature contribution to b_n from n-body scattering a term proportional to $(a/\lambda)^{n-1}$ (if there are no n-body bound states). There are terms of higher order in $1/\lambda^m$ and $(1/\lambda^m)\ln^k\lambda$, but one does not need solutions of the full n-body problem until one gets to terms of order $1/\lambda^{3n-5}$.

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Demonstration of Collisionless Interactions Between Interstreaming Ions in a Laser-Produced-Plasma Experiment

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Evidence has been obtained for collision-free momentum transfer between the ions of interstreaming plasmas in a laser-produced-plasma experiment. Diagnostics employed included fast photography, shadowgraphy, and electric potential probes. Spectroscopy provided direct evidence of momentum transfer from Doppler shifts of ion lines.

A high-density, high-temperature plasma is produced when a Q-switched laser is focused onto a small solid target. Within a few nanoseconds the thermal energy of the plasma is converted into kinetic energy of radial expansion.[1] If a low-pressure ambient gas is present, then radiation from the laser-produced plasma photoionizes the gas and the laser-produced plasma streams radially outward through the resulting ambient plasma. We report observations of momentum-transfer interactions between ions of the laser-produced plasma and ions of the ambient plasma under conditions where ordinary binary momentum-transfer collisions are negligible.

Explanation of the momentum transfer requires the existence of collective ion interactions (anomalous viscosity[2]) associated with plasma instabilities. The regime of our observations, $C_s < V < v_t$ (where V is the ion streaming velocity, C_s