The transition from the correlated pionization region to the uncorrelated pionization region occurs when $s_{a\overline{12}}s_{b\overline{12}} \cong s_{\overline{12}}M^2$ as $s_{\overline{12}}$ is increased. This allows us to define the "correlation length": As $s_{\overline{12}}$ is increased by increasing the longitudinal momenta with the transverse momenta fixed, the two particles become uncorrelated when a value of the invariant mass $s_{\overline{12}}$ is reached such that $(\vec{p}_1^{\perp}+\vec{p}_2^{\perp})^2 \ll s_{\overline{12}}$. By making use of the experimental average value of $p^{\perp}$ for the pion distributions, i.e., $\langle p^{\perp}\rangle = 300-500$ MeV, we estimate then that the two particles are correlated as long as the invariant mass is not significantly larger than 1 BeV.

For the case in which $s_{\overline{12}}$ is increased by increasing $p_1^{\perp}$ and $p_2^{\perp}$ along arbitrary directions in the transverse plane with $p_1^{\parallel}$ and $p_2^{\parallel}$ fixed, we find that the two particles are always correlated and the distribution is a function of $\cos\varphi = \hat{p}_1^{\perp}\cdot\hat{p}_2^{\perp}$, which shows a cutoff in the form

$$f_{ab}^{12} \sim \exp\{-4[p_1^{\perp 2}+p_2^{\perp 2}+2p_1^{\perp}p_2^{\perp}\cos\tfrac{1}{2}\theta + p_1^{\perp}p_2^{\perp}(1-\cos\varphi)\ln\tan\tfrac{1}{4}\varphi]\}. \tag{16}$$

The details of this paper and other related topics will be discussed elsewhere.

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# Large-Order Behavior of Perturbation Theory

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We examine the large-order behavior of perturbation theory for the anharmonic oscillator, a simple quantum-field-theory model. New analytical techniques are exhibited and used to derive formulas giving the precise rate of divergence of perturbation theory for all energy levels of the $x^{2N}$ oscillator. We compute higher-order corrections to these formulas for the $x^4$ oscillator with and without Wick ordering.

A Rayleigh-Schrödinger perturbation series is a power series $\sum A_n\lambda^n$, where $\lambda$ is the coupling constant, $n$ is the order of perturbation theory, and $A_n$ is a Rayleigh-Schrödinger coefficient. We are concerned here with the Rayleigh-Schrödinger coefficients in the perturbation expansions of the energy levels of the $N$th anharmonic oscillator, which is defined by the equations

$$[-d^2/dx^2+\tfrac{1}{4}x^2+\lambda x^{2N}2^{-N}-E(\lambda)]\psi(x)=0, \tag{1a}$$

$$\lim_{|x|\to\infty}\psi(x)=0. \tag{1b}$$

In this Letter we give formulas describing the large-$n$ behavior of $A_n$.

More precisely, we express the perturbation expansion of $E^{K,N}(\lambda)$, the $K$th energy level of the $N$th anharmonic oscillator [$E^{K,N}(\lambda)\to K+\frac{1}{2}$ as $\lambda\to 0$ along the positive real $\lambda$ axis], as

$$E^{K,N}(\lambda)\sim K+\tfrac{1}{2}+\sum_{n=1}^{\infty}A_n^{K,N}\lambda^n. \tag{2}$$

We derive here the remarkable result that for $n\to\infty$,

$$A_n^{K,N}=\frac{(N-1)2^K(-1)^{n+1}}{\pi^{3/2}K!2^n}\Gamma(nN-n+K+\tfrac{1}{2})\left\{\frac{\Gamma(2N/(N-1))}{\Gamma^2(N/(N-1))}\right\}^{nN-n+1/2}\left[1+O\left(\frac{1}{n}\right)\right]. \tag{3}$$

In addition, for the ground-state energy of the $x^4$ oscillator ($K=0$, $N=2$), we compute the first-order correction to Eq. (3):

$$A_n^{0,2}=(-1)^{n+1}\pi^{-3/2}6^{1/2}3^n\Gamma(n+\tfrac{1}{2})[1-\tfrac{95}{72}n^{-1}+O(n^{-2})]. \tag{4}$$

At the end of this Letter we give similar formulas [see Eqs. (17) and (18)] for the Wick-ordered $N=2$ oscillator. We feel that the derivations of these results constitute a significant contribution to the study of singular perturbation theory.

Much of the recent intensive interest in the nature of singular perturbation theory was generated by the discovery[1] that the Feynman perturbation expansions of the $(\varphi^{2N})_2$ quantum field theory diverge. The anharmonic oscillator [which is a $(\varphi^{2N})_1$ quantum field theory[2]] has served as a good model for research on this topic because its perturbation series are also divergent, as is evident from Eq. (3).[3] Early investigations of the anharmonic oscillator[4] were concerned with elucidating and interpreting the singularities in the complex $\lambda$ plane which cause the divergence of the power series in Eq. (2). Other work[5,6] was also concerned with summability methods [techniques which recover the eigenvalues from the $A_n^{K,N}$ and thus, in effect, "sum" the divergent series in Eq. (2)]. The results of this paper have an immediate application in summability theory. Estimates of the growth of the $A_n^{K,N}$ are used to prove that the approximants converge to the correct limits.[7,8]

Aside from the results, the new analytical techniques we shall introduce here, while not entirely rigorous, are of interest themselves. We give below two independent derivations of our results. The first derivation employs WKB techniques and establishes the first computational connection between the real- and complex-$\lambda$ dependence of the energy levels. The second derivation introduces new methods for approximating the difference equations which generate the $A_n^{K,N}$.

*Derivation 1.*—Two rigorous properties of the eigenvalues of the anharmonic oscillator are that for large $|\lambda|$, $|E^{K,N}(\lambda)|\sim|\lambda|^{1/(N+1)}$, and that $E^{K,N}(\lambda)$ is analytic in the cut $\lambda$ plane with the cut extending along the real axis from $-\infty$ to the origin. The first property is a direct consequence of the Symanzik transformation.[5] The second is a deep result obtained by Loeffel and Martin.[5] These properties imply that $F^{K,N}(\lambda)\equiv\lambda^{-1}[E^{K,N}(\lambda)-K-\frac{1}{2}]$ satisfies

$$\lim_{|\lambda|\to\infty}|F^{K,N}(\lambda)|=0, \tag{5a}$$

$$\lim_{|\lambda|\to\infty}|\lambda F^{K,N}(\lambda)|=0, \tag{5b}$$

and $F^{K,N}(\lambda)$ is analytic in the same cut $\lambda$ plane as $E^{K,N}(\lambda)$. It follows from Eqs. (5) and the analyticity properties of $F^{K,N}(\lambda)$ that $F^{K,N}(\lambda)$ obeys the dispersion relation

$$F^{K,N}(\lambda)=(2\pi i)^{-1}\int_{-\infty}^{0}(x-\lambda)^{-1}D^{K,N}(x)\,dx, \tag{6}$$

where

$$D^{K,N}(\lambda)\equiv\lim_{\epsilon\to 0}[F^{K,N}(x+i\epsilon)-F^{K,N}(x-i\epsilon)].$$

Next, we insert the relation

$$(x-\lambda)^{-1}=\sum_{n\geq 0}\lambda^n x^{-n-1}$$

into Eq. (6) and obtain the asymptotic series in Eq. (2) by interchanging orders of integration and summation. The expression for the Rayleigh-Schrödinger coefficients is thus

$$A_n^{K,N}=(2\pi i)^{-1}\int_{-\infty}^{0}dx\,x^{-n}D^{K,N}(x). \tag{7}$$

The result in Eq. (7) is *exact* because the integral exists.[5] Moreover, for large $n$ the contribution to this integral comes from a small region near $x=0$. Thus, to complete this derivation, it is sufficient to compute (approximately) the analytic continuation of the energy levels to small and negative $\lambda$. It is natural to accomplish this task using WKB theory. Indeed, zeroth-order WKB theory gives the leading behavior in Eq. (3), first-order WKB theory gives the first-order correction in Eq. (4), and so on.

However, determining $E(\lambda)$ from a WKB analysis requires extreme care because one is confronted with the problem of subdominance. In our previous work[4] the secular equation for the energy levels (an implicit relation between $E$ and $\lambda$) appears as a consistency condition when we perform asymptotic connections of approximations to the wave function $\psi(x)$ in various regions in complex $x$ space. Ordinarily, since one cannot unambiguously connect subdominant (that is, exponentially small) terms in the asymptotic expansions, the resulting secular equation is meaningless. (Fortunately, in Ref. 4, where $\arg\lambda = 270°$ for the $x^4$ oscillator, the asymptotic connections are made on Stokes lines, along which one does not distinguish between dominant and subdominant terms in the asymptotic expansions; both terms are oscillatory and do not grow or decay exponentially.) In the present problem, where $\arg\lambda = 180°$, we successfully avoid the problems of subdominance by decomposing Eq. (1a) into real and imaginary equations. This separation prevents dominant terms from eclipsing exponentially small but important terms. In addition, this separation helps to clarify the boundary conditions at $|x|=\infty$. When $\lambda$ is negative, $\psi(x)$ oscillates at $|x|=\infty$. Without decomposing Eq. (1a) into real and imaginary parts, it would be difficult to tell which linear combination of outgoing and incoming waves to use.

The details of this rather involved calculation will be presented elsewhere.[9,10] The results of zeroth-order WKB theory are

$$D^{K,N}(x)=\frac{2i2^K\pi^{1/2}}{K!}(-\tfrac{1}{2}x)^{-(K+1/2)/(N-1)}\exp\left\{\frac{-\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))(-\frac{1}{2}x)^{1/(N-1)}}\right\}. \tag{8}$$

Inserting this formula into Eq. (7) gives Eq. (3). Eq. (4) is obtained from first-order WKB theory.

*Derivation 2.*—We limit this discussion to the expansion of $E^{0,2}(\lambda)$. Let

$$E^{0,2}(\lambda)=\tfrac{1}{2}-\sum_{n=1}^{\infty}(-\lambda)^n C_n, \tag{9a}$$

$$\psi(x)=\exp(-\tfrac{1}{4}x^2)\Big[1+\sum_{n=1}^{\infty}(-\lambda)^n\sum_{j=1}^{2n}(\tfrac{1}{2}x^2)^j C_{n,j}\Big]. \tag{9b}$$

Substituting Eq. (9) into Eq. (1) gives the exact equation

$$2jC_{n,j}=(j+1)(2j+1)C_{n,j+1}+C_{n-1,j-2}-\sum_{p=1}^{n-1}C_{p,1}C_{n-p,j}, \tag{10}$$

where $C_n = C_{n,1}$ and $C_{0,0}=1$. Using Eq. (10) we determine the behavior of $C_n$ for large $n$ by making a series of approximations.

*Approximation A: successive linearization.*—$C_{n,j}$ is positive for all $n$ and $j$. Thus, in Eq. (10) the summation is small compared with the first two terms on the right-hand side. To find the leading growth of $C_n$ (zeroth order) we neglect the sum. To find the first-order $[O(n^{-1})]$ correction we retain one term ($p=1$) and insert the numerical value for $C_{1,1}$. For the $J$th order $[O(n^{-J})]$ correction we retain the first $J$ terms in the sum and replace $C_{p,1}$ by its numerical value. This gives an infinite sequence of *linear* equations which we claim give successive corrections to the asymptotic growth of $C_{n,1}$. This claim is borne out by computer to 150th order in perturbation theory. From here on we consider only the zeroth-order calculation.

*Approximation B.*—Let $C_{n,j}=D_{n,j}/j\Gamma(j+\frac{1}{2})$. $D_{n,j}$ obeys $D_{n,j}=D_{n,j+1}+[\frac{1}{2}j-\frac{3}{8}(j-2)]D_{n-1,j-2}$, $D_{1,2}=\frac{3}{8}\sqrt{\pi}$. Computer calculations verify that for $j\sim 1$, $C_{n,j+1}\gg C_{n-1,j-2}$ and for $j\sim 2n$, $C_{n-1,j-2}\gg C_{n,j+1}$. We thus can approximate this equation by

$$E_{n,j}=E_{n,j+1}+\tfrac{1}{2}jE_{n-1,j-2}. \tag{11}$$

The initial condition is determined by requiring that

$$\lim_{n\to\infty}E_{n,2n}/D_{n,2n}=1.$$

If we choose $E_{0,0}=1$, then as $n\to\infty$,

$$C_{n,1}=2\sqrt{2}\pi^{-1}E_{n,1}. \tag{12}$$

The generating functions

$$E(x)=\sum_{n\geq 1}E_{n,1}x^{n-1}/(n-1)!$$

and

$$G(x,y)=\sum\sum_{n,j\geq 1}\frac{x^n y^{j+2}E_{n,j}}{n!}$$

transform Eq. (11) into

$$\frac{\partial G}{\partial x}+\tfrac{1}{2}y^4(1-y)^{-1}\frac{\partial G}{\partial y}=y^3(y-1)^{-1}[y^2-E]. \tag{13}$$

It is remarkable that we can solve this equation even though it contains two unknowns, $G$ and $E$. The change of variable $\xi=y^{-2}-\frac{2}{3}y^{-3}$, $y=g_{1,2}(\xi)$ brings Eq. (13) into a standard form whose solution is

$$G_{1,2}(x,\xi)=\int_0^x dx'\frac{g_{1,2}^{\,2}(\xi-x+x')}{1-g_{1,2}(\xi-x+x')}[E(x')-g_{1,2}^{\,2}(\xi-x+x')]. \tag{14}$$

The subscripts 1 or 2 occur because for $y>0$, $\xi(y)$ has a two-valued inverse $g_{1,2}(\xi)$. But $g_1(\xi)=g_2(\xi)$ when $\xi=\frac{1}{3}$. Thus $G_1(x,\frac{1}{3})=G_2(x,\frac{1}{3})$, which eliminates $G$ and gives an integral equation from which we determine the dominant behavior of $E(x)$:

$$E(x)\sim\tfrac{1}{4}(\tfrac{1}{3}-x)^{-3/2}+O((\tfrac{1}{3}-x)^{-1/2}). \tag{15}$$

*Approximation C.*—Equation (15) implies that for large $n$, $E_{n,1}\sim 3^{n+1/2}\Gamma(n+\frac{1}{2})/2\sqrt{\pi}$. This result and Eq. (12) combine to give Eq. (3) (in which $K=0$ and $N=2$).

Generalizing this derivation to nonzero values of $K$ while keeping $N=2$ is relatively easy. However, values of $N>2$ give a complicated $(N-1)$-order partial differential equation in place of Eq. (13). Nevertheless, we have been able to recover all of the features of Eq. (3) except for the multiplicative constant term (independent of $n$).

*Wick ordering.*—The energy levels of the anharmonic oscillator are expressible in terms of Feynman diagrams. For $K\neq 0$, $E^{K,N}(\lambda)$ is the $K$-particle pole of the $2K$-point Green's function. The ground-state energy $E^{0,N}(\lambda)$ is the sum of all connected diagrams having no external legs.

Wick ordering a quantum field theory excludes those Feynman diagrams in which a line emerges from and returns to the same vertex. Wick ordering is necessary when the space-time dimension is two or greater because such diagrams are divergent. The anharmonic oscillator is a field theory in one-dimensional space-time and does not require Wick ordering. Nevertheless, our results for the Wick-ordered perturbation series are interesting enough to warrant presentation here.

We consider only the $N=2$ Wick-ordered oscillator, which is described by Eq. (1b) and the differential equation[9]

$$[-d^2/dx^2-\tfrac{1}{2}+\tfrac{3}{4}2+(\tfrac{1}{4}-\tfrac{3}{2}\lambda)x^2+\tfrac{1}{4}\lambda x^4-e(\lambda)]\psi(x)=0. \tag{16}$$

The perturbation expansion of the $K$th energy level, $e^{K,2}(\lambda)$, takes the form

$$e^{K,2}(\lambda)\sim K+\sum_{n\geq 1}\lambda^n a_n^{\,K,2}.$$

Equations (3) and (4) and lengthy algebra give

$$\lim_{n\to\infty}A_n^{\,K,2}/a_n^{\,K,2}=e^3=20.085\,537\cdots, \tag{17}$$

$$\lim_{n\to\infty}(A_n^{\,0,2}/a_n^{\,0,2})=e^3[1-n^{-1}-\tfrac{41}{24}n^{-2}+O(n^{-3})]. \tag{18}$$

*Theory versus computer calculations.*—By iterating difference equations such as Eq. (10) on a computer, we have calculated exactly $A_{150}^{\,0,2}$, $A_{150}^{\,1,2}$, $A_{150}^{\,2,2}$, $A_{75}^{\,0,3}$, and $a_{74}^{\,0,2}$, and find spectacular agreement with our predictions. We conclude with two such comparisons: $A_{150}^{\,0,2}(\text{computer})=7.518\,810\,301\,1\times 10^{332}$; Eq. (4) predicts $A_{150}^{\,0,2}=7.519\,480\times 10^{332}$; $A_{74}^{\,0,2}/a_{74}^{\,0,2}(\text{computer})=19.806\,55$; Eq. (18) predicts $A_{74}^{\,0,2}/a_{74}^{\,0,2}=19.807\,84$.

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