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Dual-Amplitude Analysis of Two-Particle Productions*

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We present the results on the two-particle production cross section for the process $a+b\rightarrow x_1+x_2+$anything at high energy predicted by the dual-resonance model. The distribution functions for all the kinematic regions are shown to have the desired variable dependence and properties. In particular, we find that the two-particle "correlation length" is of the order of 1 GeV^{-1}.

We would like to present in this paper the analysis of two-particle productions in high-energy inclusive hadronic reactions

$$a+b\rightarrow x_1+x_2+\text{anything}, \tag{1}$$

within the framework of the dual-resonance model. Single-particle distributions in such context have been studied previously.[1,2]

Following Mueller,[1,3] we relate the cross section for the process (1) to a discontinuity in the missing-mass variable

$$M^2=(p_a+p_b+p_{\bar{1}}+p_{\bar{2}})^2 \tag{2}$$

of the standard eight-line dual amplitude (which we denote as B_8)[4] for the process (see Fig. 1)

$$a+b+\bar{x}_1+\bar{x}_2\rightarrow\bar{a}+\bar{b}+x_1+x_2. \tag{3}$$

Then the distribution functions $f_{ab}{}^{12}$ defined by

$$\frac{E_1E_2}{\sigma_{ab}{}^{\text{tot}}}\frac{d\sigma_{ab}}{d^3p_1d^3p_2}=f_{ab}{}^{12}(s_{ab},M^2,s_{a\bar{1}},\cdots) \tag{4}$$

can be evaluated from the discontinuity of B_8 in M^2. Here $s_{ab}=(p_a+p_b)^2$, $s_{a\bar{1}}=(p_a+p_{\bar{1}})^2$, etc. denote the independent variables. These variables appear through trajectory functions via the relations $\alpha_{ab}=s_{ab}+a_{ab}$, $\alpha_{a\bar{1}}=s_{a\bar{1}}+a_{a\bar{1}}$, etc. The expression (4) is the number density for particles x_1 and x_2, and it is this formula that is conjectured to approach the limiting distributions[5] or the scaling limits[6] at high energies. The work of Mueller[3] suggests that a single leading Regge-pole exchange is sufficient to explain such limiting behavior. The dual-resonance model has the desired Regge behavior and was successful for the qualitative understanding of the single-particle distribution.[1,2]

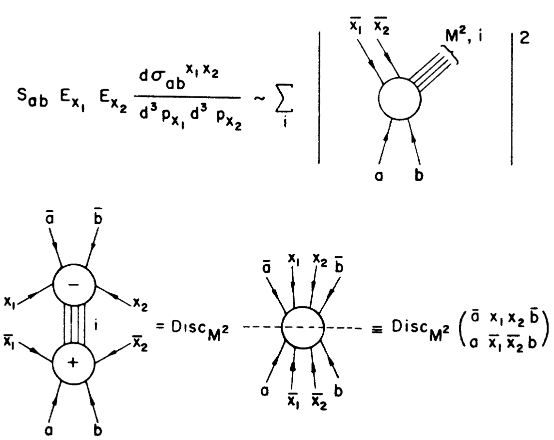

FIG. 1. The generalized optical theorem to relate the inclusive cross sections with $\text{disc}_{M^2}B_8$.

Although there are $\frac{1}{2}(8-1)!$ distinct diagrams for B_8, only $\frac{1}{2}\times 4!\times 4!$ of them, having a, $\bar{x}_1$, $\bar{x}_2$, and b adjacent, contribute to the discontinuity in M^2. Furthermore, because of the symmetry property of the dual amplitude noticed by Plahte,[7] we can reduce, following the same procedure as that of Ref. 1, the number of diagrams contributing to the discontinuity in M^2. By adopting similar phase conventions as those of Ref. 1 for the asymptotic limit along rays in complex Mandelstam variables, it is sufficient, for our present purpose, to study only the single diagram $\left(\begin{smallmatrix}\bar{a} & x_1 & x_2 & \bar{b}\\ a & \bar{x}_1 & \bar{x}_2 & b\end{smallmatrix}\right)$ (see Fig. 1). It should be remarked that this phase convention is reminiscent of any interpretation of the asymptotic behavior of the four-point dual amplitude at fixed momentum transfer or scattering angle and was shown in Ref. 1 to lead to the Mueller analysis of inclusive reactions.

The identification of (4) with the discontinuity of B_8 in M^2, or in $\alpha = M^2 + \alpha(0)$, requires an analytic continuation in α while holding all other invariant variables at certain prescribed locations.[8] Starting with the purely real integral representation for B_8 in which all the invariant variables are held below their lowest singularities, we first continue α to the region above its normal threshold and obtain the discontinuity in α by simply calculating the imaginary part. We then continue the variables α_{ab}, $\alpha_{ab\bar{1}}$, $\alpha_{ab\bar{2}}$, and $\alpha_{\bar{1}\bar{2}}$ ($\alpha_{\bar{a}\bar{b}}$, $\alpha_{\bar{a}\bar{b}1}$, $\alpha_{\bar{a}\bar{b}2}$, and α_{12}) in the discontinuity formula back to their physical region with the $+i\epsilon$ ($-i\epsilon$) prescription. The remaining invariant variables are all of the "crossed" type to M^2 and therefore do not give rise to singularities in the discontinuity. In this way, we can avoid the danger of enclosing any unwanted singularities in the analytic continuation. We note that all the invariant variables are to be treated as independent in obtaining the discontinuity which can be accomplished by treating the general nonforward limit first; and that any asymptotic kinematical constraint among the variables which may exist should *not* be invoked before the discontinuity is taken.

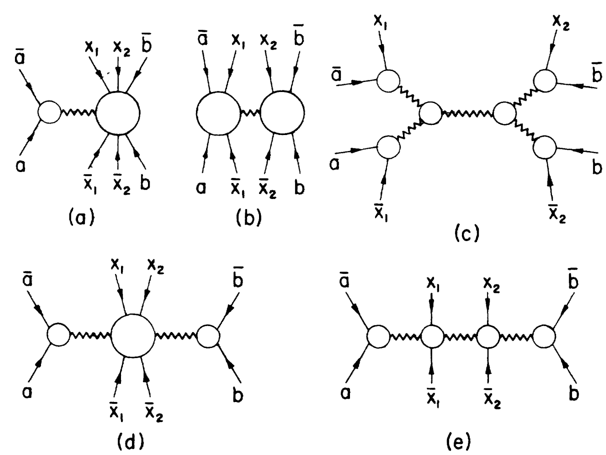

FIG. 2. Regge asymptotic limits (a) through (e) as discussed in the text for two-particle production.

We discuss the two-particle distribution functions[9] in the kinematical regions of the single and double fragmentations, the correlated and uncorrelated pionizations, and the ditriple Regge limit, when $s_{ab}\to +\infty + i\epsilon$, $s_{\bar{a}\bar{b}}\to +\infty - i\epsilon$, and $M^2 \to +\infty \pm i\epsilon$.

(1) The single fragmentation of b. This is the region [see Fig. 2(a)] in which the momenta of x_1 and x_2 are fixed in the rest frame of b so that $s_{a\bar{1}} \to -\infty$ with $s_{b\bar{2}}$, $s_{b\bar{1}\bar{2}}$, $s_{ab}/M^2 \cong 1 - s_{a\bar{1}\bar{2}}/M^2$, and

$$\frac{s_{a\bar{1}}}{M^2} \cong -\left[1 - \frac{m_1 - E_2 + p_2^{\parallel}}{E_1 - p_1^{\parallel}}\right]^{-1}$$

fixed. We obtain

$$f_b^{12}(p_1,p_2) = (M^2/s_{ab})^{\alpha_{a\bar{a}}} R_b(p_1,p_2), \tag{5}$$

where $R_b(p_1,p_2)$ is given by a fourfold integral evaluated in the rest frame of b.

(2) The double fragmentation region. Here, the momentum of, say x_1 is fixed in the rest frame of a and that of x_2 in the rest frame of b, while $s_{\bar{1}\bar{2}} \to +\infty + i\epsilon$ and $s_{12} \to +\infty - i\epsilon$ with $s_{a\bar{1}}$, $s_{b\bar{2}}$, $s_{\bar{a}1}$, $s_{\bar{b}2}$ and s_{ab}/M^2 fixed [see Fig. 2(b)], so that

$$s_{a\bar{1}\bar{2}}\, s_{b\bar{1}\bar{2}}/M^2 s_{\bar{1}\bar{2}} \cong s_{a\bar{1}\bar{2}}\, s_{b\bar{1}\bar{2}}/M^2 s_{\bar{a}a\bar{1}\bar{2}} \cong s_{a\bar{1}\bar{2}}\, s_{b\bar{1}\bar{2}}/M^2 s_{\bar{b}b\bar{2}\bar{1}} \cong 1. \tag{6}$$

We find in the c.m. system of a and b that

$$f_{ab}^{12}(p_1,p_2,s_{ab}) = f_a^{1}(p_1^{\perp}, p_1^{\parallel}/\sqrt{s_{ab}})\, f_b^{2}(p_2^{\perp}, p_2^{\parallel}/\sqrt{s_{ab}}), \tag{7}$$

where

$$f_a^{1}(p_1^{\perp}, p_1^{\parallel}/\sqrt{s_{ab}}) = \left(1 - \frac{E_1 + p_1^{\parallel}}{2E_a}\right)^{\alpha_v} \left(\frac{2E_a}{E_1 + p_1^{\parallel}}\right)^{-\alpha_{a\bar{1}} - \alpha_{\bar{a}1}} R_a^{1}(p_1^{\perp}, p_1^{\parallel}/\sqrt{s_{ab}}), \tag{8}$$

R_a^1 being the twofold integral in Eq. (3.3) of Ref. 1 for the process $a + \text{target} \to x_1 + \text{anything}$.[10] α_v denotes $\alpha_{1\bar{a}a\bar{1}}$, the dominating trajectory for describing the total cross section.

Note that (7) has the desired factorization property and that (7) and (8) are functions of the scaled c.m. or Feynman variables $(p_1{}^\perp, p_1{}^\parallel/\sqrt{s_{ab}}, p_2{}^\perp, p_2{}^\parallel/\sqrt{s_{ab}})$.

(3) The ditriple region. This is achieved when $s_{\bar{1}\bar{2}}$, s_{12}, s_{ab}/M^2, and M^2 are all large with $s_{a\bar{1}}$, $s_{\bar{a}1}$, $s_{b\bar{2}}$, and $s_{\bar{b}2}$ fixed [see Fig. 2(c)]. Then both $s_{a\bar{1}\bar{2}}/M^2$ and $s_{b\bar{1}\bar{2}}/M^2$ tend to $-\infty$. By taking these limits in (7), we find

$$f_{ab}{}^{12} = \left(\frac{M^2}{s_{ab}}\right)^{\alpha_v}\left(-\frac{s_{b\bar{1}\bar{2}}}{M^2}\right)^{\alpha_{a\bar{1}}}\left(-\frac{s_{\bar{b}12}}{M^2}\right)^{\alpha_{\bar{a}1}}\left(-\frac{s_{a\bar{1}\bar{2}}}{M^2}\right)^{\alpha_{b\bar{2}}}\left(-\frac{s_{\bar{a}12}}{M^2}\right)^{\alpha_{\bar{b}2}} h_{ab}{}^{12}, \tag{9}$$

where

$$h_{ab}{}^{12} = h_a{}^1(p_1{}^\perp, p_1{}^\parallel/\sqrt{s_{ab}})h_b{}^2(p_2{}^\perp, p_2{}^\parallel/\sqrt{s_{ab}}), \tag{10}$$

$$h_a{}^1 = \Gamma(-\alpha_{\bar{a}1})\Gamma(-\alpha_{a\bar{1}})\Gamma(1+\alpha_v)/\Gamma(-\alpha_{\bar{a}1} - \alpha_{a\bar{1}} + \alpha_v + 1). \tag{11}$$

Note that (11) is precisely the triple-Regge vertex one obtains for the production of x_1 in inclusive collisions, and that (9) is simply a product of two such triple-Regge expressions.[11]

(4) The uncorrelated pionization region. This region corresponds to $s_{a\bar{1}}, s_{b\bar{2}} \to -\infty$, and $s_{\bar{1}\bar{2}} \to +\infty$, with s_{ab}/M^2 fixed so that

$$s_{a\bar{1}\bar{2}}\, s_{b\bar{2}}/M^2 \cong m_1{}^2 + (p_1{}^\perp)^2, \quad s_{b\bar{1}\bar{2}}\, s_{a\bar{1}}/M^2 \cong m_2{}^2 + (p_2{}^\perp)^2,$$

and $s_{a\bar{1}\bar{2}}\, s_{b\bar{1}\bar{2}}/M^2 s_{\bar{1}\bar{2}} \cong 1$ [see Fig. 2(e)]. The distribution function has the correct factorization property and depends only on $p_1{}^\perp$ and $p_2{}^\perp$:

$$f_{ab}{}^{12} = f^1(p_1{}^\perp)f^2(p_2{}^\perp), \tag{12}$$

where $f^1(p_1{}^\perp)$ is the single-particle distribution function in the pionization region, i.e., Eq. (3.7) of Ref. 1. In particular, it behaves as $(p_1{}^\perp)^2 \to \infty$ like

$$f^1(p_1{}^\perp) \sim 2^{\alpha_{a\bar{a}1} + \alpha_{a\bar{a}\bar{1}} - 4\alpha_v - 2}(p_1{}^\perp)^{-2\alpha_v - 3}\exp[-4(p_1{}^\perp)^2]. \tag{13}$$

Note that the distribution function (12) has the exponential cutoff in transverse momenta of the form $\exp[-4(p_1{}^\perp)^2]\exp[-4(p_2{}^\perp)^2]$. We add that this region can also be reached smoothly from the double-fragmentation region: The transition is analogous to that between the fragmentation region and the central region in the six-point function.

(5) The correlated pionization region. This situation is reached when x_1 and x_2 are produced with a fixed invariant mass $s_{\bar{1}\bar{2}}$ at high energies [see Fig. 2(d)]. We then have $s_{ab}/M^2 \cong 1$ and

$$s_{a\bar{1}}\, s_{b\bar{1}\bar{2}}/M^2 s_{\bar{1}\bar{2}} \cong 1 + (\vec{p}_1{}^\perp + \vec{p}_2{}^\perp)^2/s_{\bar{1}\bar{2}},$$

while $s_{a\bar{1}\bar{2}}, s_{b\bar{2}} \to -\infty$. The distribution function turns out to be[12]

$$f_{ab}{}^{12} = (\alpha_{\bar{a}1}\alpha_{\bar{b}2}/\alpha_{ab})^{\alpha_v} R_{12}(\alpha_{\bar{1}\bar{2}}, p_1{}^\perp, p_2{}^\perp, \hat{p}_1{}^\perp \cdot \hat{p}_2{}^\perp), \tag{14}$$

where R_{12} is given by a threefold integral.

It is clear that the momenta x_1 and x_2 must be parallel to each other as E_1 and E_2 are increased with $s_{\bar{1}\bar{2}}$ fixed. If we consider the $p_1{}^\parallel, p_2{}^\parallel \to \infty$ limit, say with the choice of direction $\vec{p}_b \cdot \vec{p}_i = p_b p_i{}^\parallel$, while keeping $\alpha_{\bar{1}\bar{2}}$ fixed, (14) approaches gradually the limit $\alpha_{b\bar{2}} \to -\infty$ of the single fragmentation case (5). On the other hand, for fixed $\alpha_{\bar{1}\bar{2}}$, as $p_1{}^\perp$ and $p_2{}^\perp$ are increased (then, of course, $\vec{p}_1{}^\perp \cdot \vec{p}_2{}^\perp = p_1{}^\perp p_2{}^\perp \to \infty$), (14) shows a remarkable cutoff in the transverse momentum of $\vec{q} = \vec{p}_1 + \vec{p}$:

$$f_{ab}{}^{12} \sim \exp[-4\gamma_{ab}(\sqrt{\gamma_a} + \sqrt{\gamma_b})^2] \cong \exp(-4q_\perp{}^2), \tag{15}$$

where $\gamma_{ab} = s_{\bar{a}1}s_{\bar{b}2}/M^2$, $\gamma_a = s_{\bar{a}12}/s_{\bar{a}1} - 1$, and $\gamma_b = s_{\bar{b}12}/s_{\bar{b}2} - 1$. In either case, we find that the two particles behave like a single particle as E_1 and E_2 are increased so long as $s_{\bar{1}\bar{2}}$ is fixed.

As $s_{\bar{1}\bar{2}}$ is allowed to increase, however, the two particles x_1 and x_2 no longer have to come out with parallel momenta. Consider the case in which E_1 and E_2 are increased by increasing the longitudinal momenta but with the transverse momenta fixed. To allow $s_{\bar{1}\bar{2}}$ to increase would imply that the two longitudinal momenta are antiparallel to each other so that, say, x_1 approaches a while x_2 approaches b. Then (14) goes over smoothly to the factorized distribution function of the uncorrelated pionization region (12), thus showing the exponential cutoff in each transverse momentum.

The transition from the correlated pionization region to the uncorrelated pionization region occurs when $s_{a\bar{1}\bar{2}}\, s_{b\bar{1}\bar{2}} \cong s_{\bar{1}\bar{2}} M^2$ as $s_{\bar{1}\bar{2}}$ is increased. This allows us to define the "correlation length": As $s_{\bar{1}\bar{2}}$ is increased by increasing the longitudinal momenta with the transverse momenta fixed, the two particles become uncorrelated when a value of the invariant mass $s_{\bar{1}\bar{2}}$ is reached such that $(\vec{p}_1^{\perp} + \vec{p}_2^{\perp})^2 \ll s_{\bar{1}\bar{2}}$. By making use of the experimental average value of $p^{\perp}$ for the pion distributions, i.e., $\langle p^{\perp}\rangle = 300-500$ MeV, we estimate then that the two particles are correlated as long as the invariant mass is not significantly larger than 1 BeV.

For the case in which $s_{\bar{1}\bar{2}}$ is increased by increasing $p_1^{\perp}$ and $p_2^{\perp}$ along arbitrary directions in the transverse plane with $p_1^{\parallel}$ and $p_2^{\parallel}$ fixed, we find that the two particles are always correlated and the distribution is a function of $\cos\varphi = \hat{p}_1^{\perp} \cdot \hat{p}_2^{\perp}$, which shows a cutoff in the form

$$f_{ab}^{12} \sim \exp\{-4[p_1^{\perp 2} + p_2^{\perp 2} + 2p_1^{\perp} p_2^{\perp} \cos\tfrac{1}{2}\theta + p_1^{\perp} p_2^{\perp}(1-\cos\varphi)\ln\tan\tfrac{1}{4}\varphi]\}. \tag{16}$$

The details of this paper and other related topics will be discussed elsewhere.

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Large-Order Behavior of Perturbation Theory

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We examine the large-order behavior of perturbation theory for the anharmonic oscillator, a simple quantum-field-theory model. New analytical techniques are exhibited and used to derive formulas giving the precise rate of divergence of perturbation theory for all energy levels of the x^{2N} oscillator. We compute higher-order corrections to these formulas for the x^4 oscillator with and without Wick ordering.

A Rayleigh-Schrödinger perturbation series is a power series $\sum A_n \lambda^n$, where λ is the coupling constant, n is the order of perturbation theory, and A_n is a Rayleigh-Schrödinger coefficient. We are concerned here with the Rayleigh-Schrödinger coefficients in the perturbation expansions of the ener-