Simultaneous Realization of SU(3)⊗SU(3) and Dilation Symmetry*

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It is shown that if the chiral SU(3)⊗SU(3) and scale-invariant limits coincide, chiral-symmetry breaking by $(3, 3^*) \oplus (3^*, 3)$ terms provide a consistent description incorporating the results of Cheng and Dashen and of Gell-Mann, Oakes, and Renner.

In this note we discuss the implications of the recent result of Cheng and Dashen[1] within the context of the $(\underline{3}, \underline{3}^*) \oplus (\underline{3}^*, \underline{3})$-breaking model[2,3] of SU(3)⊗SU(3) symmetry.

Consider, to start with, the usual[2,3] $(\underline{3}, \underline{3}^*) \oplus (\underline{3}^*, \underline{3})$ breaking model of SU(3)⊗SU(3),

$$H = H_0 + \epsilon_0 S_0 + \epsilon_8 S_8, \quad \epsilon_8/\epsilon_0 = c, \tag{1}$$

where S_i ($i = 0, 1, \cdots, 8$) is the nonet of scalar densities which, together with an appropriate nonet of pseudoscalar densities, transforms as the $(\underline{3}, \underline{3}^*) \oplus (\underline{3}^*, \underline{3})$ representation of SU(3)⊗SU(3). The success of current algebra suggests that H_0, which is SU(3)⊗SU(3) invariant, describes a world containing a massless octet of pseudoscalar mesons. Also if SU(3)⊗SU(3) is a reasonable symmetry of nature, one expects that in the symmetry limit other quantities, like masses of baryons, coupling constants, etc., do not change appreciably from their physical values. However, the result of Cheng and Dashen[1] shows the contrary. For the value $c \simeq -1.25$ obtained by Gell-Mann, Oakes, and Renner[2] (GMOR), this result indicates that the mass of the nucleon is drastically altered[4] in the SU(3)⊗SU(3)-symmetric world, so that the physical masses of the baryons[5] seem to arise predominantly from the symmetry-breaking terms in (1). This result introduces serious problems. Consider as an example the usual or generalized Goldberger-Treiman relations which are exact consequences in the SU(3)⊗SU(3)-symmetric world. If these relations are approximately valid in nature, clearly a drastic change in the baryon mass must be compensated by an equally remarkable change in one or more of the coupling parameters that enter these relations. The approximate experimental validity of Goldberger-Treiman relations would thus be quite accidental. Note that an *ad hoc* change in the value of c is not an acceptable solution, since this would upset the pseudoscalar-meson masses, according to the analysis of GMOR.

A way out[6] of the paradox is the proposition that the SU(3)⊗SU(3)-symmetric limit coincides with the limit of scale invariance realized through the massless dilaton σ. This possibility was raised by Gell-Mann[7] and has been studied by several authors. Within the framework of the $(\underline{3}, \underline{3}^*) \oplus (\underline{3}^*, \underline{3})$ model, we suggest that this is the only simple way out. Instead of (1), we may accordingly consider the following model[8] for the energy-momentum tensor.

$$\theta_{00} = \bar{\theta}_{00} + \delta + \epsilon S, \tag{2}$$

where $\bar{\theta}_{00}$ is chiral and scale invariant, δ is chiral invariant but breaks scale invariance, and $\epsilon S = \epsilon_0 S_0 + \epsilon_0 S_8$ breaks both chiral and scale symmetries, and will be assumed to have a unique dimension d. We shall also take δ to be a c-number. For (2), one obtains the virial theorem

$$-\theta_{\mu\mu} = (4-d)\epsilon S + 4\delta. \tag{3}$$

Note that from Lorentz invariance, since $\langle 0|\theta_{\mu\mu}|0\rangle = 0$, we get $4\delta = (d-4)\langle 0|\epsilon S|0\rangle$, so that $\delta \to 0$ as $\epsilon \to 0$. Thus $\epsilon \to 0$ leads to the combined SU(3)⊗SU(3)- and scale-invariant limit. As mentioned before we have assumed that in this limit the pseudoscalar mesons as well as the scalar dilaton are massless.

Using the normalization $\langle p|-\theta_{\mu\mu}|p\rangle = M/V$, where M is the mass of the baryon described by the state $|p\rangle$ at rest, one finds for the nucleon

$$M_N/V = (4-d)\langle p|\epsilon S|p\rangle_{\text{conn.}}. \tag{4}$$

Now as stated, the result of Cheng and Dashen for the model (1) suggests that the entire mass of the nucleon arises from the symmetry-breaking terms in (1), i.e., $M_N/V \approx \langle p|\epsilon S|p\rangle$, if c is close to the GMOR value. Within the framework of the model (2), this then implies[6] that $d = 3$. More significantly, however, the result of Cheng and Dashen can now be reinterpreted for model (2) as follows. For the model (1), the Cheng-Dashen result implies a vanishing nucleon mass in the symmetry limit $\epsilon \to 0$, which leads to the difficulties mentioned above. However, for the model (2) $M_N(\epsilon \to 0) \neq 0$. To see how this arises note that the matrix element $\langle p'|\epsilon S|p\rangle$ will have a σ-pole term $\propto \epsilon/(t - m_\sigma^2)$, where $t = -(p-p')^2$.

From (4), it is clear that M_N will then have a contribution $\propto \epsilon/m_\sigma^2$ which will be finite for $\epsilon \to 0$ if $m_\sigma^2 = O(\epsilon)$. Thus if SU(3)⊗SU(3)- and scale-invariant limits are coincident as in the model (2), in general $M_N(\epsilon \to 0) \neq 0$. The importance of the σ-pole terms was first pointed out and discussed by Ellis.[9] Clearly the combined symmetry limit ($\epsilon \to 0$) would make sense if the σ-pole contribution, i.e., $M_N(\epsilon \to 0)$, nearly accounts for the entire nucleon mass, leaving the rest of ϵ-dependent terms as a small perturbation. It is easy to show from Eq. (4) that this implies[10]

$$M_N(\epsilon \to 0) = f_\sigma G_{NN\sigma} \simeq M_N, \tag{5}$$

where the coupling constants f_σ are defined through the following matrix element:

$$\langle 0|\theta_{\mu\nu}(0)|\sigma(k)\rangle = \tfrac{1}{3}(2k_0V)^{-1/2} \times f_\sigma(m_\sigma^2\delta_{\mu\nu} + k_\mu k_\nu), \tag{6}$$

and $G_{NN\sigma}$ is the $NN\sigma$ coupling constant. Note that the σ-pole term in Eq. (5) does not depend explicitly on the scale dimension d. However the value $d = 3$ is the preferred one, if as mentioned c is close to the GMOR value. One is then led to question if c for the model (2) is still given by the GMOR value. This is not *a priori* evident since one has to keep track of the σ-pole terms. To settle this question, we now turn our attention to the masses of pseudoscalar mesons in the model (2).

It has been pointed out[11] that the pseudoscalar-meson mass formula

$$m^2 = (2p_0V)\langle p|\epsilon S|p\rangle \tag{7}$$

used by GMOR for the model (1) is not generally valid for the model (2). For the model (2) the appropriate relation is instead obtained from Eq. (3):

$$2m^2 = (4-d)(2p_0V)\langle p|\epsilon S|p\rangle_{\text{conn.}}, \tag{8}$$

where we have used the normalization $(2p_0V) \times \langle p|-\theta_{\mu\mu}|p\rangle = 2m^2$ for the meson state $|p\rangle$ of mass m. We first show that if we use SU(3) parametrization for the full matrix element in Eq. (8), we reproduce the GMOR results but for $d = 2$. Thus if we assume

$$(4p_0q_0V^2)^{1/2}\langle P_i(q)|S_j(0)|P_k(p)\rangle_{\text{conn.}} = \alpha(t)\delta_{j0}\delta_{ik} + \beta(t)d_{ijk} \tag{9}$$

for $i, k = 1, \cdots, 8$, and $j = 0, 1, \cdots, 8$, we obtain from Eq. (8)

$$(m_K^2 - m_\pi^2) = -(4-d)(\sqrt{3}/4)\beta(0)\epsilon_8. \tag{10}$$

Also, note that if we consider the K_{l3} matrix element

$$(4p_0q_0V^2)^{1/2}\langle \pi^0(q)|V_\mu^{4-i5}(0)|K^+(p)\rangle = \tfrac{1}{2}\sqrt{2}[(p+q)_\mu F_+(t) + (p-q)_\mu F_-(t)], \tag{11}$$

we get for the matrix element of the divergence

$$(4p_0q_0V^2)^{1/2}\langle \pi^0(q)|\partial_\mu V_\mu^{4-i5}(0)|K^+(p)\rangle = \tfrac{1}{2}\sqrt{2}\,d(t) \tag{12}$$

with

$$d(t) = (m_K^2 - m_\pi^2)F_+(t) + tF_-(t).$$

Now using[12] $\partial_\mu V_\mu^{4-i5} = -\frac{1}{2}\sqrt{3}\,\epsilon_8 S^{4-i5}$, we obtain from Eqs. (12) and (9) the result

$$d(0) = m_K^2 - m_\pi^2 = -\tfrac{1}{2}\sqrt{3}\,\beta(0)\epsilon_8, \tag{13}$$

where in the first equation in (13) we have used $F_+(0) = 1$, the SU(3) value. Note that Eqs. (13) and (10) imply that $d = 2$, so that the mass formula (8) reduces to (7), and the analysis of GMOR goes through unaltered. The result $d = 2$ however contradicts the previous result, $d = 3$. The difficulty lies in the SU(3) parametrization in Eq. (9). Indeed, because of the existence of the low-lying σ pole, one may expect some distortions in the SU(3) result (9) near $t \simeq 0$.

Explicitly calculating the σ-pole contribution to the matrix element in Eq. (8), we then obtain

$$2m_\pi^2 = f_\sigma G_{\sigma\pi\pi} + (4-d)(2p_0V)\langle \pi|\epsilon S|\pi\rangle_{\text{n.p.}}, \tag{14}$$

$$2m_K^2 = f_\sigma G_{\sigma KK} + (4-d)(2p_0V)\langle K|\epsilon S|K\rangle_{\text{n.p.}}, \tag{15}$$

where the suffix n.p. denotes no σ pole. If we neglect all the n.p. terms and retain only the σ-pole contributions, Eqs. (14) and (15) reduce to the usual consequences of partial conservation of dilatation current (PCDC).[12] Carruthers[13] has suggested that the PCDC result for mesons should be abandoned. The reason for this in the present approach is quite transparent. Note that whereas the σ terms for the baryon mass [see Eq. (5)] are of order unity in the limit $\epsilon \to 0$, the corresponding σ terms in Eqs. (14) and (15) are of order ϵ. This is because $G_{\sigma\pi\pi}$ and $G_{\sigma KK}$ vanish in the combined symmetry limit when pions, kaons, and the σ are massless. In fact applying the hypotheses of partial conservation of axial-vector current to the matrix elements $\langle \pi|A_\mu^\pi|\sigma\rangle$ and $\langle K|A_\mu^K|\sigma\rangle$, one obtains[13] $G_{\sigma\pi\pi} \propto m_\sigma^2 - m_\pi^2$ and $G_{\sigma KK} \propto m_\sigma^2 - m_K^2$. Thus whereas the σ pole indeed dominates for the baryon masses, it gives a contribution of the same order as the n.p. terms for mesons.

Having extracted the σ contributions explicitly, we may now parametrize the n.p. matrix elements using SU(3). Using Eq. (9) only for the nonpole terms this time, and using the same symbols for economy, we then obtain

$$\begin{aligned} 2m_\pi^2 &= f_\sigma G_{\sigma\pi\pi} + (4-d)(\gamma\epsilon_0 + \beta\epsilon_8 - \sqrt{3}), \\ 2m_K^2 &= f_\sigma G_{\sigma KK} + (4-d)(\gamma\epsilon_0 - \tfrac{1}{6}\sqrt{3}\,\beta\epsilon_8), \\ \gamma &= \alpha + \sqrt{2/3}\,\beta. \end{aligned} \tag{16}$$

Note that, since the K_{l3} matrix element in Eq. (12) gets no contribution from the σ pole, Eq. (13) is still valid. From the foregoing analysis, we may also express

$$\begin{aligned} f_\sigma G_{\sigma\pi\pi} &= c_\pi(m_\sigma^2 - m_\pi^2), \\ f_\sigma G_{\sigma KK} &= c_K(m_\sigma^2 - m_K^2), \end{aligned} \tag{17}$$

where c_π and c_K are proportionality constants which can be shown[13] in general to be of order unity. Using Eqs. (16) with (13) and (17), we can now solve for m_π^2 and m_K^2 to obtain

$$m_\pi^2 = \frac{\varphi_\pi}{2+\varphi_\pi} m_\sigma^2, \quad m_K^2 = \frac{\varphi_K}{2+\varphi_K} m_\sigma^2, \tag{18}$$

where

$$\varphi_\pi = c_\pi - \tfrac{2}{3}(4-d)\left(\frac{\delta}{a+1}\right)\frac{c_K - c_\pi}{d-2+c_K}, \tag{19}$$

$$\varphi_K = c_K - \tfrac{2}{3}(4-d)\left(\frac{\delta}{a} - \frac{1}{2}\right)\frac{c_K - c_\pi}{d-2+c_\pi} \tag{20}$$

with

$$\delta = \sqrt{\tfrac{3}{2}}\,\frac{\gamma}{\beta} \text{ and } a = \frac{\epsilon_8}{(2\epsilon_0)^{1/2}}. \tag{21}$$

Note that $a = c/\sqrt{2}$, where c is the GMOR parameter in Eq. (1). For $a=-1$, the model (2) is SU(2)⊗SU(2) invariant,[2] but not scale invariant. Thus at $a=-1$, we may require $m_\pi^2 \to 0$ with $m_\sigma^2 \neq 0$. Similarly at $a=2$ we realize chimeral SU(3) symmetry[14] without scale invariance, so that we require $m_K^2 \to 0$ at $a=2$ with $m_\sigma^2 \neq 0$. These conditions give

$$\varphi_\pi(a=-1) = \varphi_K(a=2) = 0. \tag{22}$$

If we assume that d, δ, c_π, and c_K do not depend —at least do not depend sensitively—on a, we obtain from Eq. (18)–(20) and (22) the mass formulas

$$\begin{aligned} m_\pi^2 &= \frac{c_\pi(1+a)}{(d-2)a + c_\pi(1+a)} m_\sigma^2, \\ m_K^2 &= \frac{c_\pi(2-a)}{[3c_\pi - 2(2-d)]a + c_\pi(2-a)} m_\sigma^2. \end{aligned} \tag{23}$$

Solving for a and $(d-2)/c_\pi$, we get

$$a = \frac{2(m_\pi^2 - m_K^2)}{m_\pi^2 + 2m_K^2}, \tag{24}$$

$$\frac{d-2}{c_\pi} = \frac{3}{2}\,\frac{m_\sigma^2 - m_\pi^2}{m_\pi^2 - m_K^2}. \tag{25}$$

It is interesting to note that Eq. (24) gives exactly the GMOR value for a. Moreover this value of a has been obtained independently of any specific assumption on the value of d. Equations (25) and (17) lead to the determination $f_\sigma G_{\sigma\pi\pi} = \frac{2}{3}(d-2) \times (m_K^2 - m_\pi^2)$, which for $d=3$ gives $f_\sigma G_{\sigma\pi\pi} \simeq -8m_\pi^2$. This differs from the numerical result of Carruthers[13] by about a factor of 2. However, in view of the obviously crude nature of the available data, this may not be serious.

In conclusion, we wish to emphasize that the combined SU(3)⊗SU(3) and dilation symmetry broken by $(\underline{3}, \underline{3}^*) \oplus (\underline{3}^*, \underline{3})$ terms not only resolves the Cheng-Dashen paradox, but leaves the results of GMOR unchanged.

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[1]T. P. Cheng and R. Dashen, Phys. Rev. Lett. 26, 594 (1971).

[2]M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[3]S. Glashow and S. Weinberg, Phys. Rev. Lett. 20, 224 (1968).

[4]The result of Ref. (1) for the $(\underline{3}, \underline{3}^*) \oplus (\underline{3}^*, \underline{3})$ breaking model (1) implies that $\frac{1}{3}(\sqrt{2}\epsilon_0 + \epsilon_8)\langle N|\sqrt{2}S_0 + S_8|N\rangle \simeq 110$ MeV. From octet mass splitting one estimates that $\epsilon_8 \langle N|S_8|N\rangle \simeq -210$ MeV. Using $c = -1.25$ and the physical nucleon mass, one than obtains $\langle N|H_0|N\rangle \simeq -270$ MeV!

[5]Since in the SU(3)⊗SU(3) limit the vacuum is taken to be SU(3) invariant, this conclusion would follow for all baryons.

[6]This solution has independently been proposed by several authors recently. See for instance R. J. Crewther, California Institute of Technology Report No. CALT-68-295, 1971 (to be published). We have not seen this paper, but have found it referred to by J. Ellis, CERN Report No. CERN-TH-1289, 1971 (to be published). See also G. Altarelli, N. Cabibbo, and L. Maiani, to be published.

[7]M. Gell-Mann, in Lectures at the Pacific International Summer School of Theoretical Physics, University of Hawaii, 1969 (unpublished).

[8]This model has been extensively studied by J. Ellis and co-workers. See for instance Ellis, Ref. 6.

[9]J. Ellis, Phys. Lett. 33B, 591 (1970).

[10]P. Carruthers, Phys. Rev. D 2, 2265 (1970).

[11]J. Ellis, P. H. Weisz, and B. Zumino, Phys. Lett. 34B, 91 (1971).
[12]See Ref. 2. This result is also valid for the model (2).
[13]P. Carruthers, Phys. Rev. D 3, 959 (1971).
[14]S. Okubo and V. S. Mathur, Phys. Rev. Lett. 23, 1412 (1969).

Failure of the Perturbation Expansion About the SU(3)⊗SU(3) Limit*

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It is suggested that perturbation-theory corrections to the limit of spontaneously broken SU(3)⊗SU(3) symmetry fail because of the existence of a radius of convergence which is much smaller than the experimental values of the symmetry-breaking parameters. The results are compatible with SU(3) being a good symmetry for states even though the Lagrangian has approximate SU(2)⊗SU(2) symmetry.

Although the breaking of SU(3)⊗SU(3) symmetry appears to be small in some respects,[1] difficulties arise when corrections to the symmetric limit are computed in perturbation theory.[1,2] These difficulties may be traced to the way in which chiral symmetry manifests itself by the "spontaneous breakdown" mechanism and the associated occurrence of an octet of massless pseudoscalar mesons. Li and Pagels have recently shown[3] that perturbative closed loops involving the massless bosons give rise to nonanalytic (logarithmic) behavior near the origin in the symmetry-breaking coupling constant. By examination of the SU(3) σ model[4] we have found that perturbation theory fails even without considering closed loops because of the existence of a radius of convergence which is much smaller than the value of the symmetry-breaking parameter required to fit experimental data. However SU(3) symmetry breaking can be computed by perturbation theory about idealized solutions corresponding to *either* SU(3)- or SU(2)⊗SU(2)-symmetric limits. In this way one can understand explicitly how SU(3) can be a good symmetry for states while the Lagrangian is nearly SU(2)⊗SU(2) symmetric. The qualitative structure of our results suggests that the phenomenon is general.

The model, which is constructed from nonets of scalar and pseudoscalar fields σ_i, φ_i $(i=0,\cdots,8)$, is described by the Lagrangian[5]

$$\mathcal{L} = \tfrac{1}{2}\mathrm{Tr}\,\partial_\mu \mathfrak{M}^\dagger \partial^\mu \mathfrak{M} + f_1(\mathrm{Tr}\,\mathfrak{M}^\dagger\mathfrak{M})^2 + f_2\,\mathrm{Tr}\,\mathfrak{M}^\dagger\mathfrak{M}\mathfrak{M}^\dagger\mathfrak{M} + g(\det\mathfrak{M} + \mathrm{H.c.}) - \epsilon_0\sigma_0 - \epsilon_8\sigma_8. \qquad (1)$$

Here $\mathfrak{M}$ is a 3×3 matrix[6] transforming as $(\underline{3}, \underline{3}^*)$. In the limit $\epsilon_0 \to 0$, $\epsilon_8 \to 0$, $\mathcal{L}$ has SU(3)⊗SU(3) symmetry. We suppose that the couplings f_1, f_2, and g are such that in the limit the normal vacuum is unstable with respect to the Goldstone-Nambu solution.[7,8] We shall assume that in the limit $\epsilon_i \to 0$ the vacuum is SU(3) symmetric, and that the vacuum expectation value $\langle\sigma_0\rangle \neq 0$. In general we write $\langle\sigma_0\rangle \equiv \xi_0$, $\langle\sigma_8\rangle \equiv \xi_8$, and denote the SU(3)⊗SU(3)-symmetric value of ξ_0 by $\hat{\xi}_0$. We shall solve (1) in the semiclassical "tree approximation," writing $\sigma_0 = \xi_0 + \sigma_0'$, $\sigma_8 = \xi_8 + \sigma_8'$, and expanding (1) about its extremal.[9] The lowest order masses are given by the quadratic terms and the extremal condition by the elimination of the terms linear in σ_0', σ_8'.

The extremal conditions relate ϵ_0, ϵ_8 to ξ_0, ξ_8. It is convenient to use[10] the variables ξ_0 and $b \equiv \xi_8/\sqrt{2}\xi_0$ rather than ξ_0, ξ_8. We then find[11] the relations

$$\epsilon_0 = F_1(\xi_0, b), \quad \epsilon_8/\sqrt{2} = F_2(\xi_0, b), \qquad (2)$$

where the F_i are defined by the sequence of equations

$$\begin{aligned} F_1(\xi_0, b) &= \xi_0^2[\tfrac{3}{4}\xi_0 G(b) + \gamma(1-b^2)], \\ F_2(\xi_0, b) &= \xi_0^2 b[4\xi_0 H(b) - \gamma(1+b)], \\ G(b) &= 3f_1(1+2b^2) + f_2(1+6b^2-2b^3), \\ H(b) &= f_1(1+2b^2) + f_2(1-b+b^2), \end{aligned} \qquad (3)$$

and the parameter γ is $2g/\sqrt{3}$.

Before analyzing these equations to determine how ξ_0 and b depend on ϵ_0, ϵ_8 we record the numerical predictions of the model. These results are needed for the interpretation of (2) and moreover demonstrate that the model gives a reason-