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Exact Solutions of the Self-Induced Transparency Equations

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We report exact solutions of the equations of self-induced transparency which generalize the linear theory of refractive index up to theoretical peak intensities of 10^{15} W cm^{-2}. The refractive indices are real, and there is no dissipation from inhomogeneous broadening. We solve the dielectric-surface problem for these solutions. We note the possibility of regaining previous analytical solutions in a fashion which permits a slight extension and some critical examination of these.

The semiclassical equations of motion for a single, undamped two-level atom exposed to a plane-polarized electromagnetic field $\vec{E}$ of arbitrary strength can be expressed in the pseudo Bloch form

$$d\vec{r}/dt=\vec{\omega}\times\vec{r};\quad \vec{\omega}=(\omega_1,0,\omega_3). \tag{1}$$

We consider a uniform dielectric consisting of n such atoms per unit volume and no host atoms. The two-level atoms couple via the Maxwell wave equation

$$\nabla\times\nabla\times\vec{E}+c^{-2}\partial^2\vec{E}/\partial t^2=-4\pi n c^{-2}\partial^2\vec{P}/\partial t^2, \tag{2}$$

where c is the velocity of light *in vacuo*. We report exact solutions of this system of equations which contain two free parameters, namely, a frequency or reciprocal temporal length v and the field amplitude E, and which are valid on or off atomic resonance.[1] We also report corresponding rotating solutions. The solutions are distortionless in the exact sense of this, rather than in the sense of Crisp[2] or of Arrechi *et al.*,[3] i.e.,

$$\vec{E}(\vec{x},t)=\vec{E}(\vec{x}-\vec{V}t), \tag{3}$$

and exhibit self-induced transparency (SIT) first reported by McCall and Hahn.[4,5] They appear to be the natural nonlinear generalizations of linear theory and considerably extend our understanding of the SIT phenomenon. The mathematical theory bears directly on that of all the previous analytical SIT solutions.[2-8]

The notation in (1) is as follows: $\vec{r}=(r_1,r_2,r_3)$; $r_1\equiv\rho_{s0}+\rho_{0s}$; $r_2\equiv i(\rho_{s0}-\rho_{0s})$; and $r_3\equiv\rho_{ss}-\rho_{00}$. The two atomic states are labeled s (upper) and 0, and $\rho=\rho(\vec{x},t)$ is the density operator. The atom's energy spacing is $\hbar\omega_s$ and $\omega_3\equiv\omega_s$. The interaction is contained in

$$\omega_1\equiv\omega_1(\vec{x},t)\equiv-2ex_{0s}\hbar^{-1}E(\vec{x},t); \tag{4}$$

$x_{0s}=x_{s0}$ is the matrix element of the dipole operator x. The atoms are inhomogeneously broadened: In (2),

$$\vec{P}(\vec{x},t)=\hat{u}ex_{0s}\int_0^\infty g(\omega_s)r_1(\omega_s,\vec{x},t)d\omega_s, \tag{5}$$

with $g(\omega_s)$ normalized to unity; $\hat{u}$ is a unit polarization vector. Since the medium is isotropic, $\vec{E}(\vec{x},t)=\hat{u}E(\vec{x},t)$, and only the magnitudes $E(\vec{x},t)$ and $P(\vec{x},t)$ appear in the theory.

A "sharp-line" solution is one for which $g(\omega_s)=\delta(\omega_s-\omega_0)$. An important result is that if we can find a sharp-line solution of (1) and (2) which is valid on or off resonance, we can always find an inhomogeneously broadened solution[9]; and this is a SIT solution if the sharp-line solution is distortionless in the sense of (3). We have found such solutions. Exact solutions are possible because the theory described by (1) and (2) is a decorrelated one. Correlation is introduced empirically as inhomogeneous broadening. Intrinsic damping is omitted, but damping is still possible because the atomic dipoles $ex_{0s}r_1(\omega_s, \vec{x}, t)$ can rapidly dephase without total destruction by the act of radiation. Slower relaxation processes then remove the energy locked in the atomic dipoles. This process introduces an imaginary part in the complex refractive index, and this describes Beer's-law decay in linear theory. We find that such decay is explicitly quenched in the SIT phenomenon and the generalized refractive indices are wholly real.

In linear theory r_3 is constant and equal to its initial value $r_3(t_0)$. We find from (1) and (2) that the refractive index m at frequency v is given by

$$\frac{m^2(v)-1}{m^2(v)+2} = -\frac{4\pi}{3}\frac{ne^2x_{0s}^{\;2}}{\hbar}r_3(t_0)\left[\mathrm{P}\int_0^\infty \frac{g(\omega_s)2\omega_s d\omega_s}{\omega_s^{\;2}-v^2} + i\pi g(v)\right] \tag{6a}$$

(P is a principal-value integral), while for each v,

$$P(z,t) = Ce^{-\eta v z c^{-1}}\sin[v(t-t_0-\mu z c^{-1})], \quad \mu \equiv \mathrm{Re}\{m\}, \quad \eta \equiv \mathrm{Im}\{m\}; \tag{6b}$$

and again for each v, (2) means that[10]

$$E(z,t) = \tfrac{4}{3}\pi n[(m^2+2)/(m^2-1)]P(z,t) \tag{7}$$

which determines C in terms of E.

An exact solution of (1) and (2) is obtained heuristically by retaining the condition (7) instead of (2). This reduces (1) to

$$\ddot{P}(\tau) + AP(\tau) = -BP^3(\tau), \tag{8a}$$

where

$$A \equiv \omega_s^{\;2} + \omega_s r_3(\tau_0)\lambda, \quad B \equiv \tfrac{1}{2}\lambda^2 e^{-2}x_{0s}^{\;-2}, \quad \lambda \equiv \tfrac{8}{3}\pi n e^2 x_{0s}^{\;2}\hbar^{-1}(m^2+2)/(m^2-1), \quad \tau \equiv t - t_0 - mzc^{-1}. \tag{8b}$$

This heuristic argument is precisely equivalent to demanding distortionless solutions of (1) and (2). Proof of this follows best by exhibiting (1) and (2) (after first Fourier transforming to a variable ω on the time t) as the nonlinear integral equation for each atom:

$$\begin{aligned} r_1(\omega_s, z, \omega) = {}& -r_3(\omega_s, t_0)2e\omega_s x_{0s}\hbar^{-1}(\omega_s^{\;2}-\omega^2)^{-1}E(z,\omega) \\ & -(\omega_s^{\;2}-\omega^2)^{-1}(2\pi)^{-2}\int_\infty^\infty \omega_1(z,\omega')(\omega-\omega')^{-1}\int_\infty^\infty \omega_1(z,\omega'')\{\omega-\omega'-\omega''\} \\ & \times r_1(\omega_s, z, \omega-\omega'-\omega'')d\omega'' d\omega' \end{aligned} \tag{9a}$$

in which $\bar{\omega}_1(z,\omega) = -2e\hbar^{-1}x_{0s}E(z,\omega)$, the transform of $\omega_1(z,t)$. For a dielectric in a region V the most general form of E is

$$\vec{\mathrm{E}}(\vec{x},\omega) = \vec{\mathrm{E}}_{\mathrm{ext}}(\vec{x},\omega) + n\int_V F(\vec{x},\vec{x}';\omega)\cdot\vec{\mathrm{P}}(\vec{x}',\omega)d^3x', \tag{9b}$$

which is an integral of the Fourier transform of (2). [F is the tensor Green's function of this equation; $\vec{\mathrm{E}}_{\mathrm{ext}}$ is an "external" field. The linear approximation which retains only the linear term on the right-hand side of (9a) is a linear integral equation fundamental to the linear theory of refractive indices.[11,12]]

The exact solution of (8) or (9), which replaces the linearized solution (5), is the Jacobian elliptic-function solution

$$\begin{aligned} P(z,t) &= C\,\mathrm{cn}[(v\tau - K), k], \quad \tau > 0, \\ &= 0, \quad \tau < 0. \end{aligned} \tag{10a}$$

The function $\mathrm{cn}(x,k)$ is periodic in x with period $4K$, and K is an elliptic integral determined by the modulus k. We choose E and v as free parameters. Then we find $k^2 = w_1^{\;2}v^{-2}$, where the Rabi frequency $w_1 \equiv ex_{0s}E\hbar^{-1}$ and

$$\frac{m^2(v)-1}{m^2(v)+2} = -\frac{4\pi}{3}\frac{ne^2x_{0s}^{\;2}}{\hbar}\int_0^\infty \frac{g(\omega_s)r_3(\omega_s,t_0)2\omega_s}{\omega_s^{\;2}-v^2+2w_1^{\;2}}d\omega_s. \tag{10b}$$

This generalized dispersion relation exhibits power "broadening." Given E and v, m is determined by (10b), k is determined, and C is determined by (7). Then $P(z,t)$ is determined.

By making use of (9) we have also solved the problem of the dielectric half-space in which the mode (10a) runs in the region $z>0$. The input in $z<0$ proves to be the function (10a) with $m=1$ and an amplitude of, say, E_0. The amplitude E of $\vec{\mathrm{E}}(\vec{\mathrm{x}},t)$ inside the dielectric is determined by the usual transmission coefficient (corrected by the internal field): $E=\frac{2}{3}(1+m)^{-1}(m^2+2)E_0$. It mimics linear theory solely because $E(\vec{\mathrm{x}},t)$ is distortionless for both $z>0$ and $z<0$. Likewise, reflection into $z<0$ follows linear theory. Given E_0 and v we can find m, E, and k; and E_0 and v are now the free parameters.

The cn function is a Fourier series with a fundamental of frequency $\pi v/2K$ and all odd harmonics. No physical source *in vacuo* of such a perfectly phased wave group is yet available, but the group (10a) may develop inside a dielectric. Certainly this is the natural nonlinear generalization of (6b), and indeed, at resonance, $v\approx\omega_s$, and when, e.g., $w_1\leq 10^{12}$ Hz, $k\ll 1$ and (10a) is close to the sine function. Thus (10a) is the plane-polarized analog of the "spin-locked" solution[13] reported by Crisp[2] in the rotating-wave and slowly-varying-amplitude approximations. But because (10a) is an *exact* solution of (1) and (2) we now have means both of assessing these approximations and of studying odd-harmonic generation. Moreover, when $w_1\approx\omega_s$ on resonance *or* $v\approx w_1$ off resonance, $k\approx 1$, and it becomes essential to include all the odd harmonics for distortionless propagation [as Eq. (13) below shows].

It is not obvious that (10a) is a SIT solution since (10b) resonates where $v=t(\omega_s^2+2w_1^2)^{1/2}$. Then we must interpret v as $v+i\delta$ (with $\delta>0$) and the expression has both real and imaginary parts in strict analogy with (6a). However we find that the initial condition $r_3(\omega_s,t_0)$ for r_3 depends on ω_s, v, and E:

$$r_3(\omega_s,t_0)=\pm\left[1+\frac{4w_1^2(v^2-w_1^2)}{(\omega_s^2-v^2+2w_1^2)^2}\right]^{-1/2} \tag{11}$$

and

$$\frac{m^2(v)-1}{m^2(v)+2}=\mp\frac{4\pi}{3}\frac{ne^2x_{0s}^2}{\hbar}\int_0^\infty\frac{g(\omega_s)2\omega_s d\omega_s}{[(\omega_s^2-v^2)^2+4w_1^2\omega_s^2]^{1/2}}. \tag{12}$$

The final form (12) of the dispersion relation means that m is real (excluding $m^2<0$) and there is no damping. The Kramers-Kronig relations do *not* apply. We find essentially the same feature in all SIT solutions including the rotating ones: It occurs in the McCall-Hahn[5] SIT solutions[14] and their generalizations.[2,3,7] The "spin-locked" solution, for example, has $r_3=\mathrm{const}$, is trigonometric, and looks linear. However, we need $r_3(\omega_s,t_0)\approx 0$ near resonance and this precise nonlinear requirement makes the solution a SIT solution. Thus SIT seems to rely on an explicit quenching of Beer's-law decay by the ability of the perfectly phased propagating modes like (10a) to carry a measure of the information built into the initial conditions.

Unfortunately the initial condition (11) means in general that the dielectric cannot simply be prepared as an amplifier or attenuator $[r_3(\omega_s,t_0)=\pm 1]$. However, we find that if we use the averaged equations of Icsevgi and Lamb[6] in the weak-damping limit, the solution (10a) goes through and the limit of terms in (pumping/damping) replaces $r_3(\omega_s,t_0)$. This solution still requires that the phase-determining quantity $r_2(\omega_s,z,t_0)$ be carefully prepared throughout the dielectric, or we must accept a jump from $r_2(\omega_s,z,t_0)=0$ as the wave front on $t-mzc^{-1}=t_0$ penetrates into dielectric. This extreme sensitivity to initial conditions may not be a physical feature: Highly organized solutions like (10a) may ultimately impress themselves over a long enough time on a dielectric prepared in an arbitrary way—especially if forced by the matching solution *in vacuo* in the region $z<0$. In this case we can also expect to propagate chopped (and hence pulsed) solutions like (10a) in which $P(z,t)$ cuts off at $t=t_0+2\nu K$ at $z=0$ (ν is an integer).

The condition (12) reduces to $r_3(\omega_s,t_0)=\pm 1$ independent of the ω_s when, but only when, $v=w_1$. In this case $k=1$ and

$$P(\tau)=C\,\mathrm{sech}[w_1(t-mzc^{-1})],\quad E(\tau)=E\,\mathrm{sech}[w_1(t-mzc^{-1})]. \tag{13}$$

This is a pure hyperbolic-secant pulse (of area 2π) without carrier and is an exact distortionless solution of (1) and (2); it contrasts strongly with McCall's and Hahn's hyperbolic secant *envelope*. Equation (7) relates E and C with the result for $m(v)=m(w_1)$:

$$\frac{m^2(w_1)-1}{m^2(w_1)+2}=\mp\frac{4\pi}{3}\frac{ne^2x_{0s}{}^2}{\hbar}\int_0^\infty \frac{g(\omega_s)2\omega_s}{\omega_s{}^2+\omega_1{}^2}\,d\omega_s. \tag{14}$$

Thus m is real and (13) is a SIT solution. It is then of most interest at resonance, however, where $v=w_1\sim 10^{15}$ Hz; and the peak intensity and energy must be the extreme values $\sim 10^{15}$ W cm^{-2} and ~ 1 J cm^{-2}. The pulse is indeed one-half an optical cycle properly shaped to propagate as a SIT solution —the best approach to a distortionless δ-function pulse. As long as $v=w_1$, and the model is applicable, (13) and (14) apply off resonance also, however, and the pulse can therefore be of arbitrary temporal length $w_1{}^{-1}\leqslant w_s{}^{-1}$ and of arbitrary, and hence much smaller, energy. The nonlinear SIT character of the solution is now of lesser interest: $r_3(\omega_s,\tau)$ pulses (as $\pm\tanh^2 w_1\tau$) on resonance and is almost constant off resonance where $w_1{}^2\ll\omega_s{}^2$.

The new solutions (10) and (13) are of interest as natural generalizations of linear theory up to extreme powers and exceptionally short pulse lengths. As physical solutions their significance may lie in astrophysical phenomena. They are perfect in the distortionless sense of (3), however, and other approximate solutions of (1) and (2) may evolve towards them on a long enough time scale. As mathematical solutions they bear directly on the pulsed rotating SIT solutions of McCall and Hahn.[4,5] We find (a) that in vapors where n is small enough ($\sim 10^{12}$) the argument for the solutions (10a) and (13) can apply to pulse envelopes traveling at a velocity c/M which modulate carriers of frequency v traveling at velocity c/m. We find dispersion relations like (12) for both $M(v)$ and $m(v)$. On resonance this rotating solution coincides precisely with McCall and Hahn's. (b) For larger n ($\gtrsim 10^{18}$) we obtain three coupled, rather than two independent, relations for $M(v)$ and $m(v)$. In general these are incompatible and envelopes like (10a) and (13) then cease to be acceptable solutions.

We remark also that there are exact solutions of (1) and (2) which depend on the Jacobian elliptic function dn: Among these is an obvious dc solution. We shall report these solutions later.

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