Nonanalytic Elementary Excitation Spectrum at Long Wavelengths in a Bose Gas and Liquid He^4

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A rigorous microscopic analysis of a weakly interacting Bose gas at zero temperature shows that the elementary excitation spectrum $\omega(k)$ is not an analytic function of the wave vector k in the long-wavelength limit but has the form $k[c_0+c_2k^2+c_{L4}k^4\ln(1/k)+\cdots]$. It is pointed out that the structure factor $S(k)$ is not an analytic function of k and that the specific heat $C_v(T)$ at low temperatures T has the form $C_v=T^3[C_0+C_2T^2+C_{L4}T^4\ln(1/T)+\cdots]$. The extension of these results to liquid He^4 is discussed.

Many theoretical[1] and experimental[2] investigations of superfluid helium have been based on the assumption that the elementary excitation spectrum $\omega(k)$, the structure factor $S(k)$, and other functions can be expanded in a power series in the wave vector k. It is also assumed that the specific heat $C_v(T)$ can be expanded in a power series in the temperature T. Since this assumption of analyticity has not been given a firm theoretical basis for liquid He^4, a rigorous microscopic analysis of a weakly interacting Bose gas would be instructive.

We consider a weakly interacting gas of spinless mass-m bosons at density n and zero temperature. The two-body force between a pair of bosons is assumed to be short ranged and summarized by the s-wave scattering length[3] a. The small dimensionless parameter in the model is $g=4\pi am s_0$, where $s_0=(4\pi na)^{1/2}/m$ is the phonon speed in the zeroth (Bogoliubov) approximation (we take $\hbar=1$). The natural units for momentum and energy are respectively ms_0 and ms_0^2, and in these units (which will be used hereafter) $m=s_0=1$, $g=4\pi a=n^{-1}$. The elementary excitation spectrum $\omega(k)$ can be analyzed in an expansion in powers of g. The main result is that $\omega(k)$ to $O(g)$ is nonanalytic in k:

$$\omega(k)/k=c_0+c_2k^2+c_{L4}k^4\ln(1/k)+\cdots, \tag{1}$$

where

$$c_0=1+\pi^{-2}g+O(g^2),$$
$$c_2=\tfrac{1}{8}-\tfrac{17}{90}\pi^{-2}g+O(g^2), \tag{2}$$
$$c_{L4}=-\tfrac{3}{320}\pi^{-2}g+O(g^2).$$

The mechanism of propagation in the long-wavelength limit is dominated by the interaction (even though it is assumed to be weak) and is conveniently described in terms of a generalized dielectric function[4,5] $\epsilon(k,\omega)$. Because of the presence of the condensate, the elementary excitation spectrum $\omega(k)$ can be obtained from the equation $\epsilon=0$, which may be cast into the form

$$\omega^2/k^2=(v/g)[1+gF^{33}(k,\omega)], \tag{3}$$

where v is the interaction taken to be a constant, and F^{33} is the irreducible part of the longitudinal current response function[4] that contains no isolated single-interaction line.

We develop a perturbation expansion for $\omega(k)$ by expanding all terms in (3) in powers of g:

$$\omega^2=\omega^{2(0)}+g\omega^{2(1)}+O(g^2),$$
$$v/g=1+gv^{(1)}+O(g^2), \tag{4}$$
$$gF^{33}=F^{33(0)}+g[F^{33(1)}+\omega^{2(1)}\partial F^{33(0)}/\partial\omega^2]+O(g^2),$$

where all quantities on the right-hand side of (4) are evaluated at $\omega^{2(0)}$. In zeroth order, $F^{33(0)}=\frac{1}{4}k^4(\omega^2-\frac{1}{4}k^4)^{-1}$ and the well-known Bogoliubov spectrum $\omega_k^{(0)}=k(1+\frac{1}{4}k^2)^{1/2}$ follows from (3) and (4). In first order, we have from (3), (4), and the explicit form of $F^{33(0)}$

$$\omega^{2(1)}/k^2=v^{(1)}+(1+\tfrac{1}{4}k^2)^{-1}F^{33(1)}. \tag{5}$$

A common feature of the various quantities in first order is that all the relevant diagrams have a one-ring structure, as shown in Fig. 6 of Ref. 4. We evaluate the contribution of these diagrams in the standard way and find

$$F^{33(1)}=\tfrac{1}{2}(S^{(1)}+M_2^{(1)}-\mu^{(1)})+(1+\tfrac{1}{4}k^2)^{1/2}(\Lambda_+^{3(1)}+\Lambda_-^{3(1)})+\tfrac{1}{2}k(\Lambda_+^{3(1)}-\Lambda_-^{3(1)})+\tfrac{1}{2}k(1+\tfrac{1}{4}k^2)^{1/2}A^{(1)}+\tfrac{1}{4}k^2(S^{(1)}-\mu^{(1)})+F^{33r(1)}-\tfrac{1}{4}k^2n'^{(1)}, \tag{6}$$

where

$$v^{(1)}=\int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2},\quad S^{(1)}+M_2^{(1)}-\mu^{(1)}=\int\frac{d^3p}{(2\pi)^3}\lambda_p\lambda_{p+k}Q^+,$$

$$\Lambda_+^{3(1)}+\Lambda_-^{3(1)}=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}(\lambda_p-\lambda_{p+k})(\vec{p}\cdot\hat{k}+\tfrac{1}{2}k)Q^-,\quad \Lambda_+^{3(1)}-\Lambda_-^{3(1)}=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\left(\frac{\lambda_p}{\lambda_{p+k}}-1\right)(\vec{p}\cdot\hat{k}+\tfrac{1}{2}k)Q^+,$$

$$A^{(1)}=\int\frac{d^3p}{(2\pi)^3}\lambda_pQ^-,\quad S^{(1)}-\mu^{(1)}=\frac{1}{4}\int\frac{d^3p}{(2\pi)^3}\left[\frac{1-\lambda_p^2}{\lambda_p}+\left(\frac{\lambda_p}{\lambda_{p+k}}+1+2\lambda_p\lambda_{p+k}\right)Q^+\right],\tag{7}$$

$$F^{33r(1)}=\frac{1}{8}\int\frac{d^3p}{(2\pi)^3}\frac{(\lambda_p-\lambda_{p+k})^2}{\lambda_p\lambda_{p+k}}(\vec{p}\cdot\hat{k}+\tfrac{1}{2}k)^2Q^+,\quad n'^{(1)}=\frac{1}{4}\int\frac{d^3p}{(2\pi)^3}\lambda_p^{-1}(1-\lambda_p)^2,$$

and

$$Q^\pm=(\omega_k^{(0)}-\omega_{p+k}^{(0)}-\omega_p^{(0)})^{-1}\mp(\omega_k^{(0)}+\omega_{p+k}^{(0)}+\omega_p^{(0)})^{-1},\tag{8}$$

$$\lambda_p=\tfrac{1}{2}p(1+\tfrac{1}{4}p^2)^{-1/2}.\tag{9}$$

Since we are interested in the elementary excitation spectrum $\omega(k)$ in the long-wavelength limit, we expand the integrals in (7) in powers of k. As an example, consider the expansion of the integral $A^{(1)}$:

$$A^{(1)}=a_1k+a_2k^2+a_3k^3+\cdots,\tag{10}$$

where

$$a_1=-\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\frac{\lambda_p}{\omega_p^{2(0)}}=\frac{1}{2\pi^2},\qquad a_3=-\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\frac{\lambda_p}{\omega_p^{2(0)}}\frac{1}{4p^2}[5(\hat{k}\cdot\hat{p})^2-1].\tag{11}$$

It is easy to see that $a_2=0$. Although the integral a_3 converges for large p, it has a logarithmic singularity for small p. Since (10) is an expansion in powers of k/p, we cut off the logarithmic singularity at k and obtain $a_3=-(1/48\pi^2)\ln(1/k)$ + regular term. Since only the singular term is of interest, this procedure is well-defined. The other integrals in (7) may be treated in the same manner. The large-p divergences in the integrals $v^{(1)}$, $S^{(1)}+M_2^{(1)}-\mu^{(1)}$, and $S^{(1)}-\mu^{(1)}$ arise from the point nature of our model interaction and they all cancel each other. Thus the coefficients c_0, c_2, and c_{L4} in (1) are well-defined and we obtain the values quoted in (2).

An analogous calculation of the structure factor $S(k)$ to first order in g shows that $S(k)$ is not an analytic function of k about the origin and has the form

$$S(k)=(k/2c_0)[1+S_2k^2+S_{L4}k^4\ln(1/k)+\cdots].\tag{12}$$

In previous work[4] the leading temperature correction to the phonon speed c_0 was shown to be $O(T^4\ln(1/T))$. It can be shown that the leading temperature correction to the coefficient c_2 is $O(T^2\ln(1/T))$. If we assume that the thermodynamics of the Bose gas can be obtained from the elementary excitation spectrum, we find that the specific heat C_v cannot be expanded in a power series in the temperature but has the form

$$C_v=T^3[C_0+C_2T^2+C_{L4}T^4\ln(1/T)+\cdots].\tag{13}$$

The logarithmic singularity in the elementary excitation spectrum (1) [and also in (12) and (13)] arises from the singularity associated with the long-wavelength limit of the product of two single-particle propagators. Ma[6] has shown that absorption and emission processes based on this long-wavelength two-propagator singularity can support second sound at low temperatures. Therefore in a Bose gas to leading order in g, the nonanalyticity of the elementary excitation spectrum and the appearance of second sound have their basis in the long-wavelength two-propagator singularity.

We now give an argument to indicate to all orders of the perturbation expansion that $\omega(k)$ for a Bose liquid is not analytic in k. Define F^{33s} as the part of F^{33} that has an isolated pair of single-particle lines. It is straightforward to see that

$$F^{33s}\propto\int\frac{d^4p}{(2\pi)^4}(\vec{p}\cdot\hat{k}+\tfrac{1}{2}k)^2\Gamma^2\mathcal{G}(-p)\mathcal{G}(p+k),\tag{14}$$

where $p=(\vec{p},\epsilon)$, $k=(\vec{k},\omega)$, the first factor $\vec{p}\cdot\hat{k}+\frac{1}{2}k$ arises from the longitudinal current, $\mathcal{G}$ is the amplitude response function,[4] and the vertex function Γ is determined by the Ward identity $\Gamma=\partial\mathcal{G}^{-1}/\partial\omega$. Gavoret and Nozieres[7] have shown to all orders in perturbation theory that in the long-wavelength limit $\mathcal{G}^{-1}=(n/n_0mc^2)(\omega^2-c^2k^2+i\delta)$, where c is the macroscopic sound speed, and thus $\Gamma\sim k$ in the long-wavelength limit. An evaluation of (14) shows that F^{33s} contains a singular term proportional to $k^4\ln(1/k)$. From (3) we see that the appearance of a singular term in $\omega(k)$ is a general property of a Bose liquid.

Recently Phillips, Waterfield, and Hoffer[2] have analyzed their specific-heat measurements for liquid He^4 at low temperatures by fitting their data by the form $C_v(T) = AT^3 + BT^5 + CT^7$ in order to determine A and B. The present work suggests that a term proportional to $T^7 \ln(1/T)$ might also be present. Although such a term appears to be too small to measure directly, its neglect might affect the value of B determined by a fit to data in a fixed temperature interval.

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Analysis of Temperature Dependence of the Vortex Core Parameter in He II

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The results of Glaberson and Steingart on vortex-core radii at low temperatures are analyzed from the point of view of a quasithermodynamic theory and found to be in agreement with the theory.

In order to understand the nature of superfluidity in liquid He II, it is useful to examine those physical conditions under which the superfluid behavior breaks down and particularly those cases in which one can see the transition from superfluid to normal-fluid behavior. An example of such a possibility is the study of the core of the line vortices occurring in rotational motion of helium. At some distance r_c from the vortex axis (the core parameter) the superfluid properties must break down; this breakdown has its consequence, for example, on vortex-ring kinematics. By investigating the kinematics, estimates can be made of the core parameter; recently such measurements have been reported for the temperature range between 0.35 and 0.6°K.[1]

It is the purpose of this note to point out that, using only a slight extension of ordinary thermodynamics, these results can be understood without a detailed model of the core structure. Since the arguments are very general, it appears that very detailed investigations may be necessary to support any specific model for the core structure.

The argument is very simple. It assumes that there is a transition region from bulk to core, in which bulk thermodynamics is still applicable, and in which a local effective temperature may be defined. The high velocity of the superfluid flow near the axis makes it thermodynamically more economical to convert superfluid to normal fluid, and thus to have a local effective temperature higher than the surrounding bulk liquid. Arbitrarily an effective temperature some factor β above the bulk temperature is used to define the core parameter.

Using these ideas one can show[2] that the effective temperature τ at a distance r from the axis is given by the implicit relation

$$r^2 = \frac{K^2}{2C} \frac{\tau}{\tau - T} \frac{\partial \gamma}{\partial \tau},$$

where K is the vortex circulation, C is the specific heat, and $\gamma = \rho_n/\rho$.

In the temperature region in which Glaberson and Steingart have done their experiments, the