where inside the cylinder, the magnetic surface where the helical pitch of the field agrees with the perturbation is singular in that the perturbed function ψ must vanish at that layer.

K. Wakefield has calculated the graph of Fig. 1. Discussions with Dr. H. P. Furth, Dr. M. N. Rosenbluth, and Dr. P. Rutherford were elucidating.

*Work supported by the U. S. Atomic Energy Commission under Contract No. AT(30-1)-1238.

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Gradient-Induced Fission of Solitons

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(Received 5 October 1971)

A theory of nonlinear dispersive-wave propagation in inhomogeneous media is used to predict the behavior of a Korteweg–de Vries solitary wave (soliton) incident on a gradient region between two uniform regions. When the gradient induces a transition into an unstable state, the soliton fissions into a train of solitons plus, in general, an oscillatory tail. We derive formulas giving the number and amplitudes of the fission solitons. The theory is applied to surface gravity waves, magnetosonic waves, and ion-acoustic waves.

The propagation of a large class of low-frequency, long-wavelength, plane-wave disturbances in weakly nonlinear and weakly dispersive media is known[1-3] to be described by the constant-coefficient Korteweg–de Vries[4] (KdV) equation. This equation yields solitary wave solutions[4] (solitons[5]) which propagate without change of shape. If the medium contains externally imposed inhomogeneities (gradients), one expects that the solitons will no longer be stationary.[6]

In this article, we predict quantitatively the behavior of a soliton which propagates from one uniform region (1), through a gradient region, and into another uniform region (2). The scale length L of the gradient region is assumed to be small compared to the scales on which the nonlinearity and dispersion act, yet large compared to the scales of the waves themselves. The transition of a soliton from region 1 to region 2 is therefore sudden (impulsive) as far as the nonlinearity and dispersion are concerned, but slow (adiabatic) as far as the gradient is concerned.[7]

The basic steps in our analysis are as follows: firstly, to use the WKB approximation to describe the transition of the soliton from region 1 to region 2, where the soliton is no longer in a stationary state (it goes into an "excited" or "unstable" state); and secondly, to use the constant-coefficient KdV equation to describe the subsequent disintegration (fission) of the soliton—an already solved problem.[8,9] A necessarily brief abstract[10] by the authors described this method in the special case of a solitary surface gravity wave in shallow water incident upon a shoal ("shoal-induced fission of solitons"). Here we present the general result which is applicable to any type of wave for which the KdV equation is a valid asymptotic description of propagation in a uniform medium.

The relevant dimensionless parameters[11] and their relative orders which are used in the asymptotic analysis are the amplitude, dispersion, and Ursell[11] parameters, given respectively by

$$\eta = a/a_s \ll 1, \tag{1a}$$

$$\sigma = (l/l_s)^2 \gg 1, \tag{1b}$$

$$U = \eta\sigma = O(1). \tag{1c}$$

Here a is the scale amplitude of the wave, a_s is the scale amplitude of the medium, l is the scale length of the wave, and l_s is the dispersion length of the medium. In accordance with what was said above, we assume that

$$l \ll L \ll \sigma l = O(l/\eta). \tag{2}$$

The initial condition under consideration corresponds to precisely one soliton propagating in region 1, and the Ursell parameter there will have a certain value, say U_1. In the gradient region, a_s and l_s change from a_{s1} and l_{s1} to a_{s2} and l_{s2}, respectively. Since by assumption the transition occurs suddenly, the slow-acting nonlinear and dispersive effects may be neglected in the gradient region. The linear WKB approximation yields $a \propto v_s^{-1/2}$ and $l \propto v_s$, where v_s is the scale velocity (linear propagation speed) which changes from v_{s1} to v_{s2}. Combining these relations given for the ratio R of the Ursell parameter in region 2 to the Ursell parameter in region 1, we obtain

$$R = \frac{U_2}{U_1} = \left(\frac{v_{s2}}{v_{s1}}\right)^{3/2} \frac{a_{s1}}{a_{s2}} \left(\frac{l_{s1}}{l_{s2}}\right)^2. \qquad (3)$$

Direct application of the theory of Gardner, Greene, Kruskal, and Mirua[8] leads to the conclusion that if $R > 1$ the soliton in region 2 will fission into two or more solitons plus, in general, an oscillatory tail, whereas if $R \leq 1$ then there will remain only one soliton plus an oscillatory tail (which of course vanishes if $R = 1$). More precisely, the number N of fission solitons is equal to the number of bound states of the following Schrödinger eigenvalue problem[8, 9]:

$$\partial^2\psi/\partial x^2 + (\lambda + 2R\,\mathrm{sech}^2 x)\psi = 0. \qquad (4)$$

It follows[9, 10, 12] that N is the greatest integer satisfying the inequality

$$N < p(R), \qquad (5)$$

where $p(R) = \frac{1}{2}[1 + (1 + 8R)^{1/2}]$. Thus to produce N fission solitons one needs $R > R_N$, where the threshold R_N is given by $R_N = \frac{1}{2}N(N-1)$. Furthermore, the final amplitudes of the fission solitons are proportional to the eigenvalues λ_n of Eq. (4).[8] It may be shown that the ratio a_n of the nth fission soliton to the amplitude of the incoming soliton is given by

$$a_n(R) = (v_{s1}/v_{s2})^{1/2}(p - n)^2 R^{-1}, \qquad (6)$$

for $n = 1, 2, \cdots, N$. This ratio is independent of the amplitude of the incoming soliton. Equations (5) and (6) are valid for values of R greater than or less than 1, but sufficiently close to 1.

In order to apply the above theory, one only needs to know the dependence of a_s, l_s, and v_s on the physical quantity which gives rise to the gradient. We shall present several examples.

For shallow-water waves, the still-water depth $h(x)$ is taken to be nonuniform. We have $a_s = h$, $l_s = h$, $v_s = \sqrt{gh}$. Hence $R = r^{-9/4}$, where $r = h_2/h_1$, a relation already derived by Ursell.[11] Fission

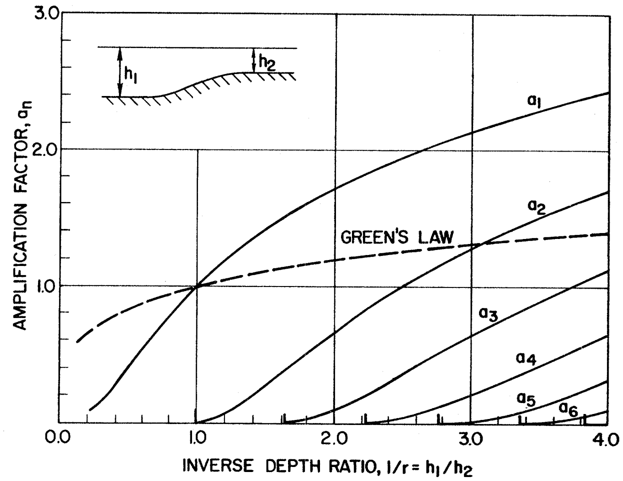

FIG. 1. Soliton amplification law for surface gravity waves [Eq. (6) with $R = r^{-9/4}$].

therefore occurs when a soliton goes from deeper water onto a shoal ($h_2 < h_1$). The phenomenon of shoal-induced fission of solitons was discovered by means of numerical experiments by Madsen and Mei.[13] They set $r = 0.5$ and observed three fission solitons of relative amplitudes 1.67, 0.75, and 0.16 plus an oscillatory tail. For $r = 0.5$, Eq. (5) predicts three fission solitons plus an oscillatory tail, and Eq. (6) gives $a_1 = 1.72$, $a_2 = 0.66$, and $a_3 = 0.10$. This is good agreement considering the approximations in the theory and the accuracy of the experiment (about ± 0.05).

Equations (5) and (6) with $R = r^{-9/4}$ were previously derived by the authors.[10] Figure 1 shows a plot of Eq. (6) for this example with a comparison to the familiar linear, nondispersive result of Green,[14] who applied what has since come to be known as the WKB approximation.

The second example is the magnetosonic soliton (Adlam-Allen pulse)[15] which is known[16] to be described by the KdV equation. Taking the magnetic field $B(x)$ to be nonuniform, we have $a_s = B$, v_s = Alfvén velocity $\propto B$, and $l_s = c/\omega_p$ (independent of B). Therefore $R = (B_2/B_1)^{1/2}$. In contrast to the previous example, fission is predicted to occur when the soliton propagates from a region of lower velocity into a region of higher velocity ($B_2 > B_1$).

The third example is the ion-acoustic soliton[17] (also known to be described by the KdV equation[1, 18]) propagating in a locally isothermal plasma with an externally imposed electronic-temperature nonuniformity $\theta(x)$. In this case, $v_s \propto \theta^{1/2}$, l_s = Debye length $\propto \theta^{1/2}$, and a_s is independent of θ since the temperature is assumed not to participate in the propagating disturbance. We then obtain $R=(\theta_2/\theta_1)^{-1/4}$. Thus fission is predicted to occur when the soliton goes from a region of higher temperature into a region of lower temperature ($\theta_2<\theta_1$), but the effect is much weaker (thresholds are higher) than shoal-induced fission of water waves.

A fission process qualitatively similar to that discussed here is expected to occur also for envelope solitions described by the nonlinear parabolic equation,[19] but that requires a separate investigation.

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Rayleigh Linewidth in SF_6 Near the Critical Point*

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In two entirely independent experiments we have measured the Rayleight linewidth for SF_6 on the critical isochore and the results are found to be in strong disagreement with previously reported measurements. Our linewidth data, after subtraction of background terms, exhibit the same critical behavior observed in other fluid systems ($\varphi=0.61\pm0.04$). Within the experimental uncertainty our results agree with linewidths calculated from the Kawasaki theory with no adjustable parameters.

The fundamental question to be answered by Rayleigh linewidth measurements on fluids near the critical point is whether or not a dynamic property such as the linewidth (the decay rate for fluctuations in the order parameter for the phase transition) behaves similarly for different systems in the same sense that the static properties are known to exhibit a similarity in critical behavior for different systems. In an effort to answer this question, there have been numerous