Helical Equilibrium of a Current-Carrying Plasma*

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By means of magnetohydrodynamic approximations, helical equilibrium configurations of current-carrying plasmas housed in a perfectly conducting cylinder were obtained. A particular solution with twisting current channels was obtained at finite amplitude. A small-amplitude equilibrium equation for arbitrary current distributions was also obtained. The presence of magnetic islands in the equilibrium configurations is apparent.

An azimuthally symmetrical toroidal equilibrium or a cylindrical equilibrium of current-carrying plasmas is well know.[1,2] These systems, however, could be subject to kink instabilities.[3] This suggests that there is a neighboring equilibrium which does not have an azimuthal symmetry. In this paper, we shall derive a finite-amplitude, helically symmetrical solution of a plasma restricted by a perfectly conducting cylinder of radius a.

The plasma is subject to a magnetohydrodynamic equilibrium condition, that is,

$$\vec{J}\times\vec{B}=\nabla p. \tag{1}$$

Cylindrical coordinates are used with $r=0$ corresponding to the axis of the cylinder. The plasma fills the interior of the cylinder. Then by introducing helical coordinates such that

$$\varphi=l\theta+k_z z, \tag{2}$$

$$B_\varphi=k_z rB_\theta-lB_z, \tag{3}$$

$$A_\varphi=k_z rA_\theta-lA_z\equiv\psi, \tag{4}$$

we obtain[4]

$$\left(k_z^2+\frac{l^2}{r^2}\right)\frac{\partial^2\psi}{\partial\varphi^2}+r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{2l}{r}\frac{l^2}{(k_zr)^2+l^2}\frac{\partial\psi}{\partial r}+\frac{2lk_zB_\varphi}{(k_zr)^2+l^2}=-\frac{\partial}{\partial\psi}\left(\frac{1}{2}B_\varphi^2\right)-\frac{dp}{d\psi}, \tag{5}$$

$$B_\varphi=B_\varphi(\psi),\quad p=p(\psi). \tag{6}$$

Here $\vec{B}$ and $\vec{A}$ represent the magnetic field and vector potential, respectively. We solve the case where $p=0$. Physically, B_φ represents the magnetic field strength parallel to the helix determined by $l\theta+k_z z=\text{const}$.

Other components of magnetic field and current density in the z direction are

$$k_z rB_z+lB_\theta=\partial\psi/\partial r, \tag{7}$$

$$B_r=-\frac{1}{r}\frac{\partial\psi}{\partial\varphi}, \tag{8}$$

$$J_z=\frac{r^2}{l^2+(k_zr)^2}\frac{1}{\mu_0}\left(\frac{\partial\psi}{\partial r}\frac{k_z}{r}-\frac{l}{r}\frac{B_\varphi}{r}\right)\frac{dB_\varphi}{d\psi}, \tag{9}$$

Thus, the solution of Eqs. (5) and (6) determines the current distribution of the plasma within the cylinder. The boundary condition is that $B_r=0$ at $r=a$, or

$$\partial\psi/\partial\varphi=0 \text{ at } r=a. \tag{10}$$

The analogy of this equation with either cylindrical or toroidal equilibrium equations is obvious. If we let $l=0$ we get a toroidal equation, and if we let $k_z=0$, we get the cylindrical equilibrium equation.

Particular solution.—By analogy with the method of Laing, Roberts, and Whipple[5] for toroidal equilibrium, we let

$$B_\varphi^2=B_0^2+k^2\psi^2 \tag{11}$$

with $|k\psi| \ll B_0$. Then Eq. (5) can be written as

$$\left(k_z^2+\frac{l^2}{r^2}\right)\frac{\partial^2\psi}{\partial\varphi^2}+r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{2}{r}\frac{l^2}{(k_zr)^2+l^2}\frac{2\psi}{\partial r}+\frac{2lk_zB_0}{(k_zr)^2+l^2}\left(1+\frac{1}{2}\frac{k^2}{B_0^2}\psi^2+\cdots\right)=-k^2\psi. \tag{12}$$

If $k^2\psi^2/B_0^2$ is small enough, Eq. (12) can be solved for arbitrary k_z and l.

However, for the purpose of understanding the general features of the above equation, it suffices to solve the case where $|k_zr| \ll l$. Then instead of Eq. (12), we get

$$\frac{l^2}{r^2}\frac{\partial^2\psi}{\partial\varphi^2}+\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi}{\partial r}+\frac{2k_z}{l}B_0=-k^2\psi. \tag{13}$$

The term B_0k_z cannot be ignored, as $|k^2\psi^2| \ll B_0^2$. Then Eq. (13) has a solution of the type

$$\psi=\frac{2k_z}{k^2l}B_0\{1+\gamma[J_0(kr)+\alpha J_l(kr)\cos\varphi]\}; \tag{14}$$

here γ and α are arbitrary constants. The boundary condition (10) requires

$$J_l(ka)=0. \tag{15}$$

The current density J_z is then

$$J_z=-\frac{2}{l\mu_0}k_zB_0\left(1+f-\frac{k_zr}{2lk^2}\frac{\partial f}{\partial r}\right), \tag{16}$$

$$f\equiv\gamma J_0(kr)+\gamma\alpha J_l(kr)\cos\varphi. \tag{17}$$

The pitch of the magnetic field line is (for $f=0$)

$$B_\theta/B_z \approx B_\theta/B_0=-k_zr/l. \tag{18}$$

That is, this finite-amplitude helical equilibrium appears when an azimuthally symmetric part ($f=0$) has the same pitch as the perturbation.

The graph of $f=\text{const}$ (Fig. 1) shows the magnetic island structure. [We note that $p=p(\psi)$, thus it follows that the plasma pressure should be constant on a given surface.] This pattern rotates helically along the axis. This pattern is similar to the one observed experimentally.[6]

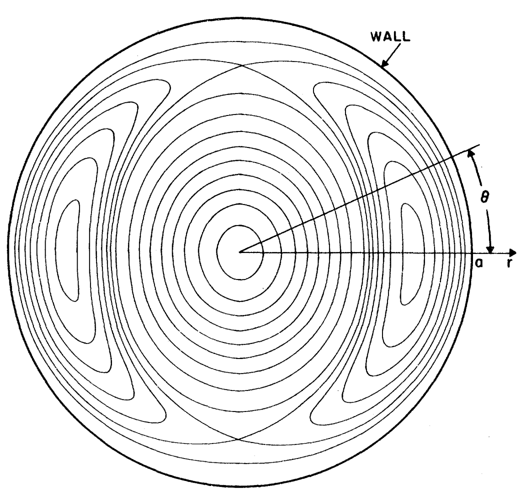

FIG. 1. A cross section of the cylindrical current channel with $l=2$ perturbation. The function $J_0(kr)+0.25\,J_2(kr)\cos 2\theta$ is shown with $J_2(ka)=0$.

Arbitrary current distribution.—For the current distribution given as $g(r)$, it is possible to determine the neighboring helical equilibrium. We now give the prescription to obtain an arbitrary solution close to the azimuthally symmetric equilibrium. We write (again for $|k_zr| \ll l$ and $B_0^2 \gg |B_\varphi^2-B_0^2|$)

$$g(r)=-\frac{1}{l\mu_0}\frac{d}{d\psi}\left(\frac{1}{2}B_\varphi^2\right).$$

Thus, choose $\psi_0=\psi_0(r)$ so that

$$\frac{1}{r}\frac{d}{dr}r\frac{d\psi_0}{dr}=-\frac{2k_zB_0}{l}+l\mu_0g(r).$$

Then define

$$h(\psi)=\psi_0^{-1}(r),$$

where ψ_0^{-1} is the inverse function. This function h can be a double-valued function of ψ. Then if we let

$$\tfrac{1}{2}B_\varphi^2=-l\mu_0\int^\psi d\psi h(\psi)+C,$$

this equation satisfies the desired current distribution, $g(r)$. Now the neighboring equilibrium solution $\psi=\psi_0+\psi_1$ can be obtained as

$$\frac{l^2}{r^2}\frac{\partial^2\psi_1}{\partial\varphi^2}+\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi_1}{\partial r}=l\mu_0\frac{\partial h}{d\psi}\bigg|_{\psi_0}\psi_1.$$

If $dJ_z/dr \neq 0$ at the point where $d\psi_0/dr=0$, ψ_1 must be zero at that point.

Conclusion.—A particular, finite-amplitude solution of helical equilibrium with a cylindrical boundary condition was obtained. Since the toroidal plasma such as Tokamak can be approximated as a straight plasma with periodic boundary condition, the nonlinear perturbation of a kink-unstable Tokamak plasma might assume the shape calculated here. We note that there exist magnetic islands indicating that the plasma transport across the magnetic field might be increased by the presence of the nonlinear perturbation.

A general prescription for finding a perturbed solution for a given azimuthally symmetric current distribution is also given. If $dJ_z/dr \neq 0$ everywhere inside the cylinder, the magnetic surface where the helical pitch of the field agrees with the perturbation is singular in that the perturbed function ψ must vanish at that layer.

K. Wakefield has calculated the graph of Fig. 1. Discussions with Dr. H. P. Furth, Dr. M. N. Rosenbluth, and Dr. P. Rutherford were elucidating.

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Gradient-Induced Fission of Solitons

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A theory of nonlinear dispersive-wave propagation in inhomogeneous media is used to predict the behavior of a Korteweg–de Vries solitary wave (soliton) incident on a gradient region between two uniform regions. When the gradient induces a transition into an unstable state, the soliton fissions into a train of solitons plus, in general, an oscillatory tail. We derive formulas giving the number and amplitudes of the fission solitons. The theory is applied to surface gravity waves, magnetosonic waves, and ion-acoustic waves.

The propagation of a large class of low-frequency, long-wavelength, plane-wave disturbances in weakly nonlinear and weakly dispersive media is known[1-3] to be described by the constant-coefficient Korteweg–de Vries[4] (KdV) equation. This equation yields solitary wave solutions[4] (solitons[5]) which propagate without change of shape. If the medium contains externally imposed inhomogeneities (gradients), one expects that the solitons will no longer be stationary.[6]

In this article, we predict quantitatively the behavior of a soliton which propagates from one uniform region (1), through a gradient region, and into another uniform region (2). The scale length L of the gradient region is assumed to be small compared to the scales on which the nonlinearity and dispersion act, yet large compared to the scales of the waves themselves. The transition of a soliton from region 1 to region 2 is therefore sudden (impulsive) as far as the nonlinearity and dispersion are concerned, but slow (adiabatic) as far as the gradient is concerned.[7]

The basic steps in our analysis are as follows: firstly, to use the WKB approximation to describe the transition of the soliton from region 1 to region 2, where the soliton is no longer in a stationary state (it goes into an "excited" or "unstable" state); and secondly, to use the constant-coefficient KdV equation to describe the subsequent disintegration (fission) of the soliton—an already solved problem.[8,9] A necessarily brief abstract[10] by the authors described this method in the special case of a solitary surface gravity wave in shallow water incident upon a shoal ("shoal-induced fission of solitons"). Here we present the general result which is applicable to any type of wave for which the KdV equation is a valid asymptotic description of propagation in a uniform medium.

The relevant dimensionless parameters[11] and their relative orders which are used in the asymptotic analysis are the amplitude, dispersion, and Ursell[11] parameters, given respectively by

$$\eta = a/a_s \ll 1, \tag{1a}$$

$$\sigma = (l/l_s)^2 \gg 1, \tag{1b}$$

$$U = \eta\sigma = O(1). \tag{1c}$$

Here a is the scale amplitude of the wave, a_s is the scale amplitude of the medium, l is the scale length of the wave, and l_s is the dispersion length of the medium. In accordance with what was said above, we assume that

$$l \ll L \ll \sigma l = O(l/\eta). \tag{2}$$

The initial condition under consideration corresponds to precisely one soliton propagating in re-