satisfies all Poisson bracket relations (6)–(9).

As an example, consider

$$H = (\vec{p}_1{}^2 + \vec{p}_2{}^2)/2m, \quad (20)$$

a conspicuously *nonrelativistic* Hamiltonian. Then

$$M^2 = (1/4m^2)(\vec{p}_1{}^2 + \vec{p}_2{}^2)^2 - \vec{P}^2, \quad (21)$$

$$= (1/16m^2)[\vec{P}^2 + (\vec{p}_1 - \vec{p}_2)^2]^2 - \vec{P}^2. \quad (22)$$

We obtain

$$\vec{p}_1 - \vec{p} = \vec{n}[4m(M^2 + \vec{P}^2)^{1/2} - \vec{P}^2]^{1/2}, \quad (23)$$

where

$$\vec{n} \equiv (\sin\alpha \cos\beta, \sin\alpha \sin\beta, \cos\alpha). \quad (24)$$

From (10) and (24) we readily obtain $\vec{p}_1$ and $\vec{p}_2$ as functions of $\vec{P}$, M^2, α, and β, which shall henceforth be considered as our new canonical momenta P_k.

We now take

$$F = \vec{q}_1 \cdot \vec{p}_1 + \vec{q}_2 \cdot \vec{p}_2, \quad (25)$$

where $\vec{p}_1$ and $\vec{p}_2$ have to be expressed in terms of $\vec{P}$, M^2, α, and β. Obviously, F is rotationally symmetric; so $\vec{Q} = \partial F/\partial \vec{P}$ is a vector.

We shall not pursue this example since it is devoid of any physical significance. It clearly shows, however, that the fulfillment of the Poisson bracket relations (1)–(9) does not guarantee Lorentz invariance unless further restrictions are imposed on the canonical variables.

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Solutions of the Coupled Einstein-Maxwell Equations Representing the Fields of Spinning Sources

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A wide class of exact solutions of the stationary Einstein-Maxwell equations characterized by a flat "background" three-space is obtained. The solutions can be interpreted as the external gravitational and electromagnetic fields of one or more spinning sources with unit specific charge in stationary configuration.

Recent theoretical and experimental results[1-3] have underlined the necessity of studying exact solutions of the general-relativistic field equations. Realistic models of gravitational collapse and its final state are expected to account for the implications of the presence of angular momentum and the interaction of gravitating systems with electromagnetism. Mainly because of the complicated nonlinear structure of the field equations, the only explicit final-state model meeting these requirements is the Kerr solution and its counterpart, vacuum except possibly for electromagnetic fields, found by Newman and his coworkers.[4] The present Letter is aimed at a radical remedy of this situation. A variety of physically meaningful stationary solutions of the Einstein-Maxwell field equations will be constructed below, including the Kerr-Newman metric with specific charge $|e|/m$ equal to unity.

We adopt the form of the general stationary line element[5]

$$d\tau^2 = -f^{-1}g_{ik}\,dx^i dx^k + f(dt + \omega_i\,dx^i)^2 \qquad (i, k, \cdots = 1, 2, 3), \quad (1)$$

where the metric field variables are functions of the spacelike coordinates x^i and where $t = x^0$ (we are using units chosen so that the velocity of light $c=1$ and the gravitational constant of Einstein $k=2$). It should be noted that the coordinate transformations

$$t' = t + t_0(x^k), \tag{2a}$$

$$x^{i'} = x^{i'}(x^k) \tag{2b}$$

are still permissible once (1) has been adopted.[5] In the usual vector notation referring to the (3×3) "background" metric g_{ik} and its inverse g^{ik}, the equations of interacting stationary gravitational and electromagnetic fields take the form[6]

$$(\nabla - \vec{G})\cdot\vec{G} = \vec{H}\cdot\vec{H} - \vec{G}\cdot\vec{G}, \tag{3a}$$

$$\nabla\times\vec{G} = \vec{H}^*\times\vec{H} - \vec{G}^*\times\vec{G}, \tag{3b}$$

$$(\nabla - \vec{G})\cdot\vec{H} = \tfrac{1}{2}(\vec{G} - \vec{G}^*)\vec{H}, \tag{3c}$$

$$\nabla\times\vec{H} = -\tfrac{1}{2}(\vec{G} + \vec{G}^*)\times\vec{H}, \tag{3d}$$

$$R_{ik} + G_iG_k^* + G_i^*G_k - H_iH_k^* - H_i^*H_k = 0. \tag{3e}$$

Here $R_{ik} \equiv R_{ir}{}^r{}_k$ is the Ricci tensor obtained from g_{ik}; the complex, three-component vectors $\vec{G}$ and $\vec{H}$ are constructed from the metric variables and the four-potential $A_\mu = (A_0, \vec{A})$:

$$\vec{G} = (2f)^{-1}(\nabla f + i\vec{\Omega}), \tag{4a}$$

$$\vec{H} = f^{-1/2}\nabla\Phi, \tag{4b}$$

where the complex function Φ and the real vector $\vec{\Omega}$ are defined as

$$\mathrm{Re}\Phi = A_0, \tag{5a}$$

$$\mathrm{Im}\Phi = -f(\nabla\times\vec{A} + \vec{\omega}\times\nabla A_0), \tag{5b}$$

$$\vec{\Omega} = -f^{-2}\nabla\times\vec{\omega}. \tag{5c}$$

The integrability conditions for $\mathrm{Im}\Phi$ follow from the field equations. The gauge freedom

$$A_\mu'(x^i) = A_\mu(x^i) + \partial_\mu\lambda \tag{6}$$

and (5) imply that Φ is determined up to an additive complex constant. Conversely, given Φ and $\vec{\Omega}$, Eqs. (5b) and (5c) yield $\vec{A}$ and $\vec{\omega}$, apart from gradient terms corresponding to the time-scaling and gauge freedoms (2a) and (6).

The condition $\vec{H}=0$ characterizes sourceless gravitational fields, while $\vec{G}=\vec{H}=0$ holds for the flat Minkowski space. Thus, roughly speaking, $\vec{H}$ represents the Maxwell field and $\vec{G}$ may be associated with the gravitational field in the present formalism. Let us consider now the fields for which the background space with metric g_{ik} is flat. Solution of the field equations (3) yields

$$\vec{G} = -\nabla\ln\psi, \quad \vec{H} = -e^{i\delta}\bar{\psi}^{1/2}\psi^{-3/2}\nabla\psi \tag{7}$$

with the arbitrary real constant δ (duality angle), and complex harmonic function ψ satisfying the flat-space Laplace's equation

$$\Delta\psi = 0. \tag{8}$$

Relying on definitions (4) and (5), all field quantities can be generated directly from the complex harmonic ψ according to

$$\begin{aligned} &f^{-1} = \bar{\psi}\psi, \quad \Phi = e^{i\delta}\psi^{-1}, \quad A_0 = \mathrm{Re}\Phi, \\ &\omega_{i;j} - \omega_{j;i} = -i\epsilon_{ijk}(\psi\bar{\psi}^{;k} - \bar{\psi}\psi^{;k})\sqrt{g}, \\ &A_{i;j} - A_{j;i} = f^{-1}\epsilon_{ijk}\mathrm{Im}\Phi^{;k}\sqrt{g} + \omega_i A_{0;j} - \omega_j A_{0;i}, \end{aligned} \tag{9}$$

where the semicolon in a suffix indicates covariant derivatives referring to g_{ik}. Integration constants have been absorbed in f and Φ, using the available freedom in the choice of the variables.

Our results can be summarized in the form of a compact recipe for constructing exact solutions of the field equations (3): (i) Choose any two solutions of the usual flat-space Laplace's equation and combine them to form the complex function ψ. (ii) Calculate the field variables using the explicit formulas (9), further the coordinate and gauge freedoms (2) and (6), respectively. Asymptotically well-behaved solutions result if we add a further rule: (iii) Let ψ be chosen so that for large values of the radial coordinate r it behaves like

$$\psi = 1 + \frac{M}{r} + iJ\frac{\cos\theta}{r^2} + \eta, \tag{10}$$

where η stands for real terms of order r^{-2} as well as for imaginary ones of order r^{-3}, and θ is the polar angle. M and J denote the total mass and angular momentum of the source, respectively.

The class of solutions obtained this way is the stationary generalization of the static class discovered independently by Majumdar[7] and Papapetrou.[8] In fact, if we choose ψ to be real by use of Eqs. (2a) and (9) we can set $\omega=0$, which is a property characteristic of static fields.

Taking $\eta=0$, identically, in (10), we obtain a particular stationary solution with

$$\begin{aligned} &f^{-1} = (1 + M/r)^2 + (J\cos\theta/r^2)^2, \\ &\Phi = f(1 + M/r - iJ\cos\theta/r^2), \end{aligned} \tag{11}$$

and with the line element

$$d\tau^2 = -f^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] + f\left[dt - J\frac{\sin^2\theta}{r}\left(2 + \frac{M}{r}\right)d\varphi\right]^2. \tag{12}$$

The appearance of a true singularity at $r=0$ together with the far-field approximation show that this field is produced by a source centered at the origin, having an electric charge and magnetic moment equal to the mass M and angular momentum $\vec{J}=(J,0,0)$, respectively (in units of $c=k/2=1$). By reversing the sign of ψ, a solution with opposite electromagnetic moments is obtained. We can also change the signs of $\mathrm{Re}\psi$ and $\mathrm{Im}\psi$ separately, and thus manipulate the electric and magnetic moments independently. To reverse the direction of angular momentum, in addition, the reflection $\varphi \to -\varphi$ should be performed in (12).

The line element (12) is axially symmetric. We can, however, easily construct solutions *without* any spatial Killing symmetry. As an example, let us transform the source of (11) and (12) from the origin to an arbitrary position and orientation and take the superposition of several ψ's thus obtained. It is clear that the resulting field, in the general case, will not show any spatial symmetry. Writing

$$\psi=\psi_1(\vec{r},\vec{r}_1,\vec{J}_1)+\psi_2(\vec{r},\vec{r}_2,\vec{J}_2)+\cdots, \tag{13}$$

the corresponding field contains singularities at $\vec{r}_1$, $\vec{r}_2$, $\cdots$. If we associate the location of the sources with singular regions, we conclude that the field generated by (13) is produced by several charged, spinning bodies held in equilibrium configuration by their balanced gravitational and electromagnetic interactions.

By direct calculation it can be established that the Kerr-Newman field possesses a flat background space if and only if $e^2=m^2$, that is, if the total charge, apart from a sign, is equal to the mass of the source. Using spheroidal coordinates (R,Θ) given by

$$\begin{gathered}[(R-m)^2+a^2]\sin^2\Theta=r^2\sin^2\theta,\\ (R-m)\cos\Theta=r\cos\theta,\end{gathered} \tag{14}$$

the appropriate generating function is

$$\psi=1+m/(R-m+ia\cos\Theta), \tag{15}$$

where the real constant a, following Kerr, denotes the angular momentum per unit mass of the source. We transform to polar coordinates[9]:

$$\psi=1+m/(r^2+2iar\cos\theta-a^2)^{1/2}. \tag{16}$$

The singularity is located at the imaginary point $\vec{r}_0$ $(r=ia,\theta=0)$. We can evidently move the source by performing a rotation followed by a translation in the three-space. As a result, we have the field of a Kerr-Newman object with arbitrary spin orientation and location. Care should be taken here of the transformation properties of the complexified position vector[10] $\vec{r}_0$. Taking a combination of several displaced Kerr-Newman fields, we get solutions representing the equilibrium state of more sources. Note that the Kerr-Newman field with $e^2=m^2$, as given by (15), contains a nonsingular event horizon[11] at $R=m$ only if $a=0$, that is, in the static Reissner-Nordström limit.

While the Kerr-Newman solution has a fixed moment structure,[12] for the present class of solutions no restriction is made on the moments, except that any given choice of the mass moments determines, up to a sign, the electric and magnetic moments and *vice versa*.

It seems not hopeless that the generalized Israel conjecture[1,2] can be proved (or, very unlikely, disproved) to be valid, at least for the present class of stationary fields. Work on this is encouraging also since eventually the "restricted" proof may suggest a way of attacking the general problem.

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