Gravitational Radiation from a Particle Falling Radially into a Schwarzschild Black Hole*

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We have computed the spectrum and energy of gravitational radiation from a "point test particle" of mass m falling radially into a Schwarzschild black hole of mass $M \gg m$. The total energy radiated is about $0.0104mc^2(m/M)$, 4 to 6 times larger than previous estimates; the energy is distributed among multipoles according to the empirical law $E_{2^l\text{-pole}} \approx (0.44m^2c^2/M)e^{-2l}$; and the total spectrum peaks at an angular frequency $\omega = 0.32c^3/GM$.

In view of the possibility that Weber may have detected gravitational radiation,[1] detailed calculations of the gravitational radiation emitted by fully relativistic sources are of considerable interest. Three such calculations have been published in the past: waves from pulsating neutron stars, by Thorne[2]; waves from rotating neutron stars, by Ipser[3]; and waves from a physically unrealistic collapse problem (important for the points of principle treated), by de la Cruz, Chase, and Israel.[4] To these, this paper adds a fourth calculation: the waves emitted by a body falling radially into a nonrotating black hole. This calculation is particularly important for two reasons: (i) It is the first accurate calculation of the spectrum and energy radiated by any realistic black-hole process (though upper limits on the energy output have been derived by Hawking[5]); (ii) Weber's events involve such high fluxes that black holes are more attractive as sources than are neutron stars.

A first analysis of the radial-fall problem was done by Ruffini and Wheeler[6] with a simple idealization: The particle's motion is derived from the Schwarzschild metric, but its radiation is calculated using the flat-space linearized theory of gravity. This scheme yielded a total energy radiated of $0.00246mc^2(m/M)$ and a spectrum peaked at an angular frequency $0.15c^3/GM$. Zerilli,[7] using the formalism of Regge and Wheeler,[8] gave the mathematical foundations for a fully relativistic treatment of the problem. Unfortunately, Zerilli's equations are sufficiently complicated as to make a calculation of the energy release inaccessible to analytic means.

We have used Zerilli's equations (corrected for errors in the published form), and by numerical techniques we have (i) computed the wave form of gravitational radiation, (ii) evaluated the amplitude of this wave asymptotically at great distances, and (iii) used this amplitude to compute the outgoing wave intensity in units of energy per unit frequency per unit of solid angle.

Zerilli describes the 2^l-pole component of gravitational waves by a radial function $R_l(r)$ which is a combination of the Fourier transform of metric perturbations in the Regge-Wheeler formalism. The function $R_l(r)$ satisfies the remarkably simple Zerilli wave equation ($G = c = 1$)

$$d^2R_l/dr^{*2} + [\omega^2 - V_l(r)]R_l = S_l, \qquad (1)$$

with

$$r^* = r + 2M\ln(r/2M - 1). \qquad (2)$$

$V_l(r)$ is an "effective potential" defined by

$$V_l(r) = (1 - 2M/r)[2\lambda^2(\lambda+1)r^3 + 6\lambda^2Mr^2 + 18M^2r + 18M^3]/r^3(\lambda r + 3M). \qquad (3)$$

Here, $\lambda = \frac{1}{2}(l-1)(l+2)$ and $S_l(r)$ is the 2^l-pole component of the source of the wave. We are interested in the particular case of a particle initially at infinity ($t = +\infty, r = +\infty$) and falling radially into a Schwarzschild black hole ($t = +\infty, r = 2M$). For this simple case the source may be written as

$$S_l(r) = \frac{4M}{\lambda r + 3M}(l + \tfrac{1}{2})^{1/2}\left(1 - \frac{2M}{r}\right)\left[\left(\frac{r}{2M}\right)^{1/2} - \frac{i2\lambda}{\omega(\lambda r + 3M)}\right]e^{i\omega T(r)}. \qquad (4)$$

Here $t = T(r)$ describes the particle's radial trajectory giving the time as a function of radius along the

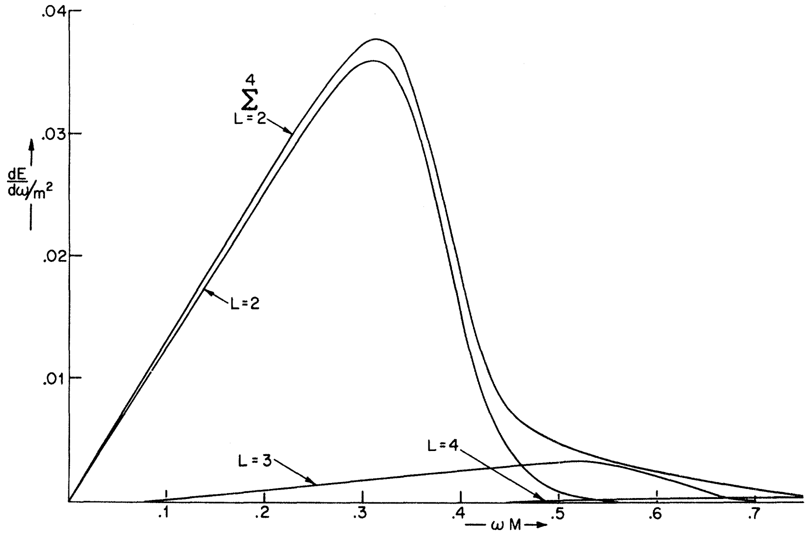

FIG. 1. Spectrum of gravitational radiation emitted by a test particle of mass m falling radially into a black hole of mass M (in geometrical units $c=G=1$).

geodesic

$$T(r)=-\frac{4}{3}\left(\frac{r}{2M}\right)^{3/2}-4\left(\frac{r}{2M}\right)^{1/2}+2\ln\left\{\left[\left(\frac{r}{2M}\right)^{1/2}+1\right]\left[\left(\frac{r}{2M}\right)^{1/2}-1\right]^{-1}\right\}. \qquad (5)$$

The effect of gravitational radiation reaction on the particle's motion is therefore ignored. This is justified by the final result: The total energy radiated, of order m^2c^2/M, is negligible compared to the particle's final kinetic energy, of order mc^2. Equation (1) is solved with boundary conditions of purely outgoing waves at infinity and purely ingoing waves at the Schwarzschild radius:

$$R_l \to \begin{cases} A_l^{\text{out}}(\omega)\exp(i\omega r^*) & \text{as } r^*\to+\infty, \\ A_l^{\text{in}}(\omega)\exp(-i\omega r^*) & \text{as } r^*\to-\infty. \end{cases} \qquad (6)$$

The energy spectrum is determined by Zerilli's formula,

$$\left(\frac{dE}{d\omega}\right)_{2^l\text{-pole}}=\frac{1}{32\pi}\frac{(l+2)!}{(l-2)!}\omega^2|A_l^{\text{out}}(\omega)|^2.$$

Two distinct methods were used to calculate $A_l^{\text{out}}(\omega)$: (i) direct integration of Eq. (1) with a numerical search technique to determine both the phase and the amplitude of the outgoing wave at infinity that would give a purely ingoing wave at the black-hole surface [details of this analysis done by two of us (M.D. and R.R.) will be published elsewhere]; (ii) integration by a Green's-function technique (see Zerilli[7]). This method allows the coefficient A_l^{out} to be computed directly as an integral involving the source term Eq. (4) and certain homogeneous solutions to Eq. (1).

All these calculations gave results in agreement within a few percent. The results are summarized in Figs. 1–3. The total energy radiated away in gravitational waves is

$$E_{\text{total}}\approx 0.0104mc^2(m/M). \qquad (8)$$

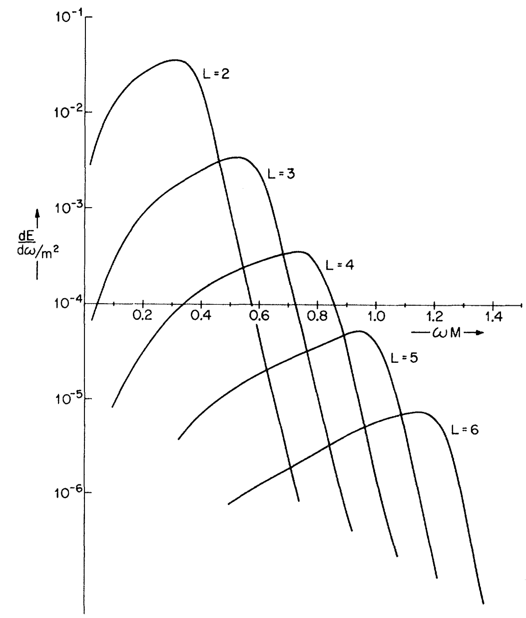

FIG. 2. Details of the spectrum of gravitational radiation integrated over all angles for the lowest five values of the multipoles.

This is about 6 times larger than Zerilli's estimate of the energy and 4 times larger than the estimate of Ruffini and Wheeler based on a purely linearized theory. The spectrum of the outgoing radiation is the superposition of a series of overlapping peaks, each peak corresponding to a certain multipole order l. Roughly 90% of the total energy is in quadrupole ($l=2$) radiation and 9% is in octupole ($l=3$). The total energy contributed by each multipole falls off quickly with l obeying the empirical relation (Fig. 3)

$$E_{2^l\text{-pole}} \approx (0.44m^2c^2/M)e^{-2l}. \qquad (9)$$

The spectrum shown in Fig. 1 is for the energy integrated over all angles. An observer at a particular angle θ from the path of the particle's fall will see a slightly different spectrum because of the different angular dependence of the various 2^l-poles. For example, a pure 2^l-pole has the angular dependence

$$(dE/d\Omega)_{2^l\text{-pole}} = E_{2^l\text{-pole}}[(l-2)!/(l+2)!]\{2\partial_\theta^2 Y_0^l(\theta,\varphi) + l(l+1)Y_0^l(\theta,\varphi)\}^2. \qquad (10)$$

As shown in Fig. 2, the energy contribution of progressively higher multipoles peaks at progressively higher angular frequencies, with the approximate relation

$$\omega(E_{2^l\text{-pole}}, \text{peak}) \approx \{c^2[V_l(r)]_{\max}\}^{1/2} \approx lc^3(27)^{-1/2}/GM \text{ for large } l. \qquad (11)$$

Each energy peak may be interpreted as due to a train of gravitational waves produced by 2^l-pole normal-mode vibrations of the black hole which the in-falling body excites (see Press[9]). Averaging over angular factors and summing the various l's, one finds that the total spectrum is peaked at $\omega = 0.32c^3/GM$, and falls off at higher ω according to the empirical law

$$dE_{\text{total}}/d\omega \sim \exp(-9.9GM\omega/c^3) \qquad (12)$$

Aside from the interesting details of our numerical results, the very fact that they are well behaved is important. Extrapolation of the flat-space linearized theory indicates that only a small fraction of a test body's rest mass $[\sim(m/M)mc^2]$ should be converted to wave energy during "fast" parts of its orbit (parts with durations $\sim GM/c^3$). It has been an open question whether this estimate holds in the region of strong fields very near the black hole. If the estimates were wrong, our results would have been divergent, with either increasing l or increasing ω. In fact, our results are strongly convergent.

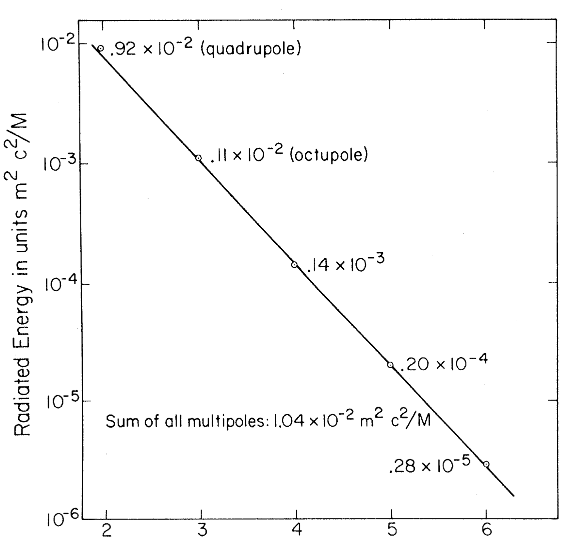

FIG. 3. Total energy radiated by each multipole. Quadrupole radiation contributes 90% of the total energy, and higher multipoles contribute progressively smaller amounts. The solid line is a plot of conste^{-2l}, an empirical fit to the data.

The other side of the coin is equally important: Although our computation verifies the linearized theory's dimensional estimate, it shows that a completely relativistic treatment can give quantitative amounts of gravitational radiation substantially larger than the linearized theory would predict.[10]

This research was performed independently and simultaneously at Caltech and Princeton, using different integration techniques but arriving at identical results. We thank Kip S. Thorne and Jayme Tiomno for helpful suggestions.

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[1]J. Weber, Phys. Rev. Lett. 22, 1320 (1969), and 24, 276 (1970), and 25, 180 (1970).

[2]K. S. Thorne, Atrophys. J. 158, 1 (1969).

[3]J. R. Ipser, Astrophys. J. 166, 175 (1970).

[4]V. de la Cruz, J. E. Chase, and W. Israel, Phys. Rev. Lett. 24, 423 (1970).

[5]S. Hawking, to be published.

[6]R. Ruffini and J. A. Wheeler, in *Proceedings of the Cortona Symposium on Weak Interactions*, edited by L. Radicati (Accademia Nazionale Dei Lincei, Rome, 1971).
[7]F. J. Zerilli, Phys. Rev. D 2, 2141 (1970).
[8]T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063 (1957).
[9]W. H. Press, to be published.
[10]M. Davis, R. Ruffini, and J. Tiomno, to be published, will give further details on the intensity and pattern of radiation for this problem and more general particle orbits.

Possible Evidence for the Existence of Antimatter on a Cosmological Scale in the Universe

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We present some initial results of a detailed calculation of the cosmological γ-ray spectrum from matter-antimatter annihilation in the universe. The similarity of the calculated spectrum with the present observations of the γ-ray background spectrum above 1 MeV suggests that such observations may be evidence of the existence of antimatter on a large scale in the universe.

The question of the existence of antimatter on a cosmological scale is one of the most basic problems of physics and cosmology. Recently it has taken on added interest because of the suggested possible role of antimatter in galaxy formation and the evolution of the universe.[1-3] It has long been recognized that the most promising way to search for evidence of antimatter on a cosmological scale is to attempt to observe γ rays of cosmic origin which would be produced by the decay of neutral pions arising from matter-antimatter annihilation. In order to determine the annihilation origin of such γ rays, one must first calculate the γ-ray spectrum which would be produced by such annihilations so that the calculated spectrum may be compared with observational data on the γ-ray background spectrum for possible identification.

Because of their possible significance as evidence of the existence of antimatter on a cosmological scale, we present here the results of such a detailed calculation, the details of which will be presented in full in a future paper.

Our calculations are in accord with the suggestions proposed by the models of Harrison[2] and Omnès.[3] These models assume that the universe consists of equal amounts of matter and antimatter which were separated into distinct regions at the earliest stages in the big-bang model of the universe. Pugét[4] has shown that in the energy region of observational interest discussed here ($E_\gamma \gtrsim 1$ MeV), where the red shift z of the origin of the γ-ray background is $\lesssim 100$, we can assume that annihilations take place on the boundaries of colliding regions of matter and antimatter so that the total annihilation rate is proportional to $(1+z)^6$. We use the commonly defined cosmological parameters H_0 and Ω, where H_0 is the Hubble constant and Ω is the ratio of the average atomic matter density in the universe to the critical density $n_c \simeq 10^{-5}$ cm^{-3} needed to close the universe gravitationally. We also define the quantity ξ which denotes the mean fraction of the total atomic density interacting at the boundaries between the regions of matter and antimatter. The atomic densities n_p and $n_{\bar{p}}$ are each proportional to $(1+z)^3$, and it is assumed that on the average $n_p = n_{\bar{p}} = \Omega n_c$. Thus we can define a mean density of interacting matter (or antimatter) $\tilde{n}^{\rm int}$ given by $\tilde{n}^{\rm int} = \xi \Omega n_c$.

The cross sections for H-$\overline{\rm H}$, p-$\bar{p}$, p-$\overline{\rm H}$, and H-$\bar{p}$ annihilation as a function of temperature for "low temperatures" ($T < 10^{11}$ K) have been included in the calculation. They are based on the recent calculations of Morgan and Hughes.[5]

The annihilation cross sections as a function of relative velocity can be expressed in various energy regions as a power law of the form

$$\sigma_A(v) = \sigma_i (v/c)^{-\delta_i}, \qquad (1)$$

where $i = 1, 2, 3$ and where

$$\sigma_1 = 4.8\times 10^{-26}\ \text{cm}^2, \quad \delta_1 = 1, \quad \text{for } 10^{11}\ \text{K} \lesssim T \lesssim 10^{13}\ \text{K};$$

$$\sigma_2 = 2.2\times 10^{-27}\ \text{cm}^2, \quad \delta_2 = 2, \quad \text{for } 10^{4}\ \text{K} \lesssim T \lesssim 10^{11}\ \text{K};$$