*Work supported in part by National Science Foundation Grant No. GP-11728.

[1]M. D. Girardeau, J. Math. Phys. 4, 1096 (1963), and 11, 681 (1970), and 12, 1799 (1971).

[2]R. H. Stolt and W. E. Brittin, Phys. Rev. Lett. 27, 616 (1971).

[3]A. Y. Sakakura, Phys. Rev. Lett. 27, 822 (1971).

[4]M. D. Girardeau, in Lectures in Theoretical Physics, Summer Institute for Theoretical Physics, University of Colorado, Boulder, Colorado, 1971 (to be published).

[5]S. Tani, Phys. Rev. 117, 252 (1960).

[6]We assume that the system consists of finite numbers of nuclei and electrons enclosed in a box of *finite* volume.

[7]To simplify the algebra we assume here that the nuclei are all protons, and in fact that the system is composed of free protons, free electrons, and hydrogen atoms. The method is, however, easily generalized.

[8]This simplifies the commutation relations between these fields and the bound-atom operators A_α and $A_\alpha^\dagger$ which will presently be defined.

[9]To obtain a complete set, the continuum atomic states would have to be included.

[10]We use the same symbol to denote the vacuum state in $\mathcal{S}$ and that in $\mathcal{Q}$; no confusion should arise thereby.

[11]The generator F is closely related to the operator Q occurring in the "quasichemical equilibrium" theory of J. M. Blatt and T. Matsubara, Progr. Theor. Phys. 20, 553 (1958).

[12]The derivation involves summation of the multiple commutator expansion to infinite order or solution of appropriate "equations of motion."

[13]A more systematic treatment of these constraints by use of a projection operator, as in Refs. 1 and 4, will be discussed elsewhere, along with details of the present derivation.

[14]The terms $\psi^\dagger\psi^\dagger\psi^\dagger\psi^\dagger aa$ and their conjugates, representing complete real or virtual breakup of colliding atoms into their constituent protons and electrons and the inverse recombination process, do not yet appear in the approximation described by Stolt and Brittin. In our atom-atom interaction matrix element $(\alpha\beta|H|\gamma\delta)$, the direct Coulomb and Coulomb-exchange contributions appear with opposite signs, as in the theory of ferromagnetism, whereas they both appear with positive signs in the Stolt-Brittin matrix element. The same matrix element already contains kinetic energy-exchange coupling contributions in the approximation of Stolt and Brittin, whereas in our treatment they do not appear until the expansion (10) is carried to the next order. Note, however, that the terms exhibited in (10) already involve selective summation of the multiple commutator expansion to infinite order. There are some additional sign differences which can be removed by a change of phase of the electron field operators and hence have no observable effect.

Kubo Conductivity of a Strongly Magnetized Two-Dimensional Plasma

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The Kubo formula is used to evaluate the bulk electrical conductivity of a two-dimensional guiding-center plasma in a strong dc magnetic field. The particles interact only electrostatically. An "anomalous" electrical conductivity is derived for this system, which parallels a recent result of Taylor and McNamara for the coefficient of spatial diffusion.

That a two-dimensional magnetized plasma can exhibit apparently "anomalous" transport properties largely because of its two-dimensional character has recently been shown by Taylor and McNamara.[1] In a guiding-center plasma with only electrostatic interactions, they have calculated the thermal-equilibrium spatial diffusion coefficient and have shown it to fall off as $|\vec{B}|^{-1}$, where $\vec{B}$ is the (constant) dc magnetic field, confirming a now classical conjecture of Bohm.[2] Additional results for the kinetic theory of a two-dimensional plasma, both with and without the guiding-center approximation, have been reported by Vahala and Montgomery.[3] This Letter describes a calculation of the *electrical conductivity* for the Taylor-McNamara model.[1] It, too, is shown to fall off as the inverse first power of B.

In this model, the ith charge moves under the influence of the electric field $\vec{E}(\vec{x}, t)$ according to $\vec{v}_i(t) = c\vec{E}(\vec{x}_i, t)\times\vec{B}/B^2$, where $\vec{B}$ is constant and normal to the plane of the motion. Both $\vec{E}$ and $\vec{v}_i(t) = d\vec{x}_i/dt$ are two-dimensional vectors normal to $\vec{B}$. $\vec{E}$ is generated from Poisson's equation according to

$$\nabla\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 4\pi\sum_i \frac{e_i}{l}\delta(\vec{x}-\vec{x}_i). \tag{1}$$

The ith charge e_i is a rod of length l, with e_i and

$l\to\infty$. The sum in Eq. (1) runs over N rods of charge $+e$ and N of charge $-e$ inside a square boundary of edge L. The initial values of the $\vec{x}_i$ are understood to be statistically distributed according to the canonical distribution of Gibbs.

The static conductivity for thermal equilibrium is calculated by the Kubo[4] technique. The Liouville equation for the system[3] is

$$\frac{\partial D}{\partial t}+c\sum_{\substack{i,j\\ i\neq j}}\frac{\vec{B}\times\partial\tilde{\varphi}_{ij}/\partial\vec{x}_i}{B^2}\cdot\frac{\partial}{\partial\vec{x}_i}D\equiv\left(\frac{\partial}{\partial t}+H_0\right)D=0, \quad (2)$$

where $D=D(\vec{x}_1,\vec{x}_2,\cdots,t)$ is the probability distribution in the phase space $\vec{x}_1,\vec{x}_2,\cdots$ of the guiding centers; $\tilde{\varphi}_{ij}=\varphi_{ij}/e_i$, and the interaction potential of charges i and j is $\varphi_{ij}=-(2e_ie_j/l)\ln|\vec{x}_i-\vec{x}_j|$.

The thermal-equilibrium solution of the Liouville equation is the Gibbs distribution,

$$D^{(0)}=\eta\exp(-\sum_{i<j}\varphi_{ij}/kT),$$

where η is a normalizing constant. If a weak external electric field $\vec{E}_0e^{\epsilon\tau}$ is added, Liouville's equation becomes

$$(\partial/\partial t+H_0)D^{(1)}=-H_1D^{(0)}\equiv-[c(\vec{E}_0e^{\epsilon\tau}\times\vec{B})/B^2]\cdot\textstyle\sum_i\partial D^{(0)}/\partial\vec{x}_i. \quad (3)$$

In this expression, $D^{(1)}$ is the linear response of the probability distribution function generated by the external field according to Liouville's equation. The solution for $D^{(1)}$ can be obtained from Eq. (3) since it is linear. The limit $\epsilon\to0$ is taken at the end of the calculation.

From the expression for $D^{(1)}$, we can calculate[4] expectation values of any dynamic variable $\vec{A}(\vec{x}_1,\vec{x}_2,\cdots)$. We indicate an ensemble average with respect to the Gibbs distribution by $\langle\rangle$. By $\langle\rangle'$ we mean an ensemble average with respect to the perturbed probability distribution $D^{(0)}+D^{(1)}$. If $\langle\vec{A}\rangle$ vanishes in the absence of the external field, the following relation is readily established:

$$\langle\vec{A}\rangle'=\lim_{\epsilon\to0}\int_{-\infty}^{0}d\tau\, e^{\epsilon\tau}\left\langle\vec{A}(\vec{x}_1,\vec{x}_2,\cdots)\sum_j\frac{e_j\vec{v}_j(\tau)}{kT}\right\rangle\cdot\vec{E}_0. \quad (4)$$

Here, kT is the temperature in energy units that characterizes the Gibbs distribution represented by $\langle\rangle$.

To determine the conductivity tensor from Eq. (4) we must choose for $\vec{A}$ a function that will make $\langle\vec{A}\rangle'$ equal to the electric-current volume density in the bulk of the plasma, $\langle\vec{j}\rangle$ (we must be careful not to apply the result at the edge of the plasma, however). Such a function is

$$\vec{A}=\sum_i\frac{e_i\vec{v}_i}{lL^2}=\sum_i\frac{ce_i\vec{E}(\vec{x}_i,t)\times\vec{B}}{B^2lL^2},$$

where the sum runs over all charges of both signs. The conductivity tensor is then

$$\overleftrightarrow{\sigma}=\lim_{\epsilon\to0}\sum_{i,j}\frac{e_ie_j}{L^2lkT}\int_0^\infty d\tau\, e^{-\epsilon\tau}\langle\vec{v}_i(0)\vec{v}_j(\tau)\rangle. \quad (5)$$

The contributions of the terms with $i\neq j$ are negligible for large L compared to those with $i=j$, which are all equal. To a good approximation, (5) is therefore

$$\overleftrightarrow{\sigma}=2\frac{N}{L^2}\frac{e^2}{lkT}\int_0^\infty\langle\vec{v}(0)\vec{v}(\tau)\rangle d\tau=2\frac{N}{L^2}\frac{e^2}{lkT}\frac{c^2}{B^4}\vec{B}\times\int_0^\infty d\tau\,\langle\vec{E}(0)\vec{E}(\tau)\rangle\times\vec{B}. \quad (6)$$

The autocorrelations in (6) are those for any single particle, and the autocorrelation $\langle\vec{E}(0)\vec{E}(\tau)\rangle\equiv Q(\tau)\overleftrightarrow{I}$ is what was calculated in connection with spatial diffusion in Ref. 1. Evaluating the cross products in (6), we see that $\overleftrightarrow{\sigma}$ is diagonal and that $\overleftrightarrow{\sigma}=\sigma\overleftrightarrow{I}$, where the conductivity

$$\sigma=2\frac{N}{L^2}\frac{e^2}{lkT}\frac{c^2}{B^2}\int_0^\infty Q(\tau)\,d\tau.$$

Inserting the thermal-equilibrium value for $\int_0^\infty Q\,d\tau$ from Ref. 1,[5] the result is

$$\sigma=\frac{n_0e^2}{lkT}\left\{\sqrt{2}\frac{ckT}{eB}\left[\frac{1}{4\pi n_0\lambda_D{}^2}\ln\left(\frac{L}{2\pi\lambda_D}\right)\right]^{1/2}\right\}. \quad (7)$$

Here, $n_0\equiv N/L^2$ and $\lambda_D{}^{-2}\equiv8\pi n_0e^2/lkT=(\text{Debye length})^{-2}$. Equation (7) is written as it is to show the proportionality between σ and the Bohm[2] diffusion coefficient. As with Taylor and McNamara's spatial diffusion coefficient, a weak (logarithmic) divergence with increasing size of the system is apparent in Eq. (7). (This is a phenomenon which has also been noted recently in connection with transport properties in neutral two-dimensional gases.[6])

The question of the degree of applicability of the two-dimensional results to three-dimensional plasmas remains open. But in those circumstances, if any exist, in which a strongly magnetized plasma *can* be considered to move in two dimensions, Eq. (7) provides a possible route for estimating a substantially higher resistivity than the conventional expression based on binary encounters in three dimensions.

We give for reference purposes a formal expression for the ac conductivity as well:

$$\sigma_{ac}(\omega)=2\frac{N}{L^2}\frac{e^2}{lkT}\frac{c^2}{B^2}\int_0^\infty e^{+i\omega\tau}Q(\tau)\,d\tau, \qquad (8)$$

though the actual value of $\sigma_{ac}(\omega)$ depends upon numerical evaluation of the definite integral in (8), which we have done and shall report in detail elsewhere. Equation (7) is a satisfactory approximate expression for frequencies less than $4\pi ec/\lambda_D^2Bl$.

Equations (7) and (8) are examples of what Kubo terms a "generalized Einstein relation," though it was not *a priori* obvious that such relations would exist for the guiding-center plasma.

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[1]J. B. Taylor and B. McNamara, Phys. Fluids 14, 1492 (1971).

[2]D. Bohm, in *The Characteristics of Electrical Discharges in Magnetic Fields*, edited by A. Guthrie and R. K. Wakerling (McGraw-Hill, New York, 1949).

[3]G. Vahala and D. Montgomery, University of Iowa Report No. 71:14 (to be published).

[4]R. Kubo, J. Phys. Soc. Jap. 12, 570 (1957).

[5]We obtain a result a factor of $\sqrt{2}$ larger than Taylor and McNamara's Eq. (29), a matter of minor importance.

[6]See, for example, T. E. Wainwright, B. J. Alder, and D. M. Gass, Phys. Rev. A 4, 233 (1971), and references therein; M. H. Ernst, E. H. Hauge, and J. M. J. van Leeuwen, Phys. Rev. Lett. 25, 1254 (1970); J. R. Dorfman and E. G. D. Cohen, Phys. Rev. Lett. 25, 1257 (1970); K. Kawasaki, Phys. Lett. 32A, 379 (1970).

Metastable Thermodynamic States Near the Critical Point of He^3†

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We report the first measurements of the specific heat of a metastable, superheated, pure liquid (He^3). The specific heat shows no evidence of any singularity near the onset of metastability. We have also observed the lifetimes of metastable states as the He^3 is cooled at constant density. For densities near the critical density, lifetimes decrease abruptly in an extremely narrow, reproducible temperature interval. This may indicate the onset of an "intrinsic" instability.

We report the first measurements of the specific heat (C_v) of a liquid in states which are metastable with respect to formation of bubbles of vapor. Since these measurements were made near the critical point of He^3, they are of interest from several points of view. First, the "equilibrium" properties of the metastable states can be compared with those model equations of state which may be continued into a metastable region within the liquid-vapor coexistence curve. Such comparison will be particularly valuable near the singularity at the critical point. Secondly, the observation of lifetimes of metastable states will enable the testing of theories which predict the rate of homogeneous nucleation of instability by local thermodynamic fluctuations. Finally, it is conceivable that observations on metastable states may demonstrate the existence of a frequently conjectured spinodal line, where "intrinsic" or large-scale instability occurs.

Since measurements of the "equilibrium" properties of metastable states of fluids are quite rare,[1,2] we now describe our apparatus and qualitative observations. We also mention the difficulty encountered in attempting to extrapolate the "linear model" equation of state[3] to the observed limit of stability. Detailed comparison of the specific-heat data with model equations of state will be described elsewhere. Detailed comparison of lifetimes with nucleation calculations will require additional experiments, some of which are in progress.

An idealized sketch of our calorimeter is shown in Fig. 1. Most of the 1.06-cm^3 sample of He^3 was contained in an accurately horizontal cylindrical volume 3.97 cm in diameter and 0.087 cm high. A vertical fill hole 0.72 cm long and 0.035 cm in diameter leads up from the main volume