are only slightly perturbed [inequality (4)]. Some particles will always be trapped in the potential well of the wave and will oscillate back and forth with a period[2] $t_{\text{trap}} \cong 2\pi(m/ekE_0)^{1/2}$. Our analysis will be correct if the wave damps appreciably in a time less than t_{trap}, i.e., if $\gamma t_{\text{trap}} \gtrsim 1$. If we neglect the β terms in Eq. (11), we find $\gamma t_{\text{trap}} = 4\pi(\alpha/B_0)(m/m_p)^{1/2}$ times the ordinate values of Fig. 1; and the criterion is not satisfied if $\alpha/B_0 \cong 1$. But, as discussed by Stix,[2] randomization of the particle trajectories by "collisions" with other waves in a broad spectrum of waves may effectively extend the time over which our calculation is valid.

We conclude by mentioning a few cases where nonlinear Landau damping of Alfvén waves may be important. Large-amplitude Alfvén waves have been observed[12-14] in the solar wind, and damping of these waves in the manner described could represent an important energy source to the solar wind.[15] Similarly, Alfvén waves originating at the surfaces of stars which have hydrogen convection zones might play a significant role in driving stellar winds and producing stellar mass loss. Finally, Alfvén waves almost certainly exist in supernova remnants and in the galaxy.

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Dynamic Structure of Vortices in Superconductors*

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A complete set of time-dependent Ginzburg-Landau equations are solved to find the current, charge, and field distributions when the vortices in a superconductor are forced into motion by a transport current. We find that in general the local electric field is not proportional to the local magnetic field and backflow is an essential feature.

A new consistent quantitative description of the motion of vortices in a superconductor as forced by a transport current is obtained which incorporates backflow as an essential feature. Previous solutions of time-dependent equations found simply a rigid translation of the vortices perpendicular to a uniform transport current.[1-3] This previous work did not generally include simultaneously all the necessary constraints including conservation of charge and the Coulomb force. By considering a complete set of equations we have found a complete solution including the local field and charge and current distributions near the upper critical field H_{c2}. We show that in general a simple low-velocity Lorentz-transformation–type relation between the local electric and magnetic fields and the translational velocity v of the vortices, $\vec{E}(r) = -\vec{v}\times\vec{B}(r)$, is not possible, although such a relation is true for the spatially averaged fields, $\langle\vec{E}\rangle = -\vec{v}\times\langle\vec{B}\rangle$.

The simplest complete set of time-dependent equations for a superconductor has been derived from microscopic theory by Gor'kov and Éliashberg[4] for a gapless superconducting alloy containing a high concentration of paramagnetic impurities[5]:

$$\gamma(\partial/\partial t+i2e\psi)\Delta+\xi^{-2}(|\Delta|^2-1)\Delta+(\nabla/i-2e\vec{A})^2\Delta=0, \tag{1}$$

$$\vec{j}=\sigma(-\nabla\psi-\partial\vec{A}/\partial t)+\mathrm{Re}[\Delta^*(\nabla/i2e-\vec{A})\Delta](4\pi\lambda^2)^{-1}, \tag{2}$$

$$\rho=(\psi-\varphi)/4\pi\lambda_{\mathrm{TF}}^2. \tag{3}$$

Here γ is the inverse of the diffusion constant. ψ may be identified as the electrochemical potential divided by the electronic charge e. Δ is the order parameter reduced by its equilibrium value in the absence of fields. ξ and λ are the temperature-dependent coherence length and penetration depth, respectively. σ is the normal-state conductivity, and λ_{TF} is the Thomas-Fermi screening length. The relation between these parameters and microscopic ones may be obtained by comparing our expressions with those of Ref. 4 (we have set $\hbar=c=1$). Adding the Maxwell equations coupling the scalar and vector potentials φ and $\vec{A}$ to the charge and current densities ρ and $\vec{j}$, a complete set of equations is obtained.

The static solution near H_{c2} has been obtained by Abrikosov taking the z direction parallel to the magnetic field and choosing the gauge where the vector potential is entirely in the y direction:

$$\Delta=\sum_n C_n\exp[-eB_0(x-kn/2eB_0)^2+ikny], \tag{4}$$

$$A_y=B_0x-\int_0^x\delta|\Delta|^2dx/4e\lambda^2\equiv A_0+\delta A, \tag{5}$$

$$B_z=B_0-\delta|\Delta|^2/4e\lambda^2\equiv B_0+\delta B. \tag{6}$$

The notation $\delta|\Delta|^2$ means the local deviation from the average, $\delta|\Delta|^2=|\Delta|^2-\langle|\Delta|^2\rangle$. The constant $B_0=\langle B\rangle$ appears in Eq. (4) so that there is exactly one flux quantum contained in each lattice cell. For the simple experimental arrangement of a planar sample very thin compared with its lateral dimensions, which is oriented perpendicular to the magnetic field, the demagnetization coefficient is unity, and B_0 equals the externally applied field.

The moving solution is obtained as a perturbation of this static solution. First we note that λ_{TF} is of the order of the Fermi wavelength (a few angstroms) and is much shorter than ξ and λ. From $\nabla\cdot\vec{E}=4\pi\rho$ and Eq. (3) it follows that $\psi=\varphi$ except for corrections $O(\lambda_{\mathrm{TF}}^2/\xi^2)$ or $O(\lambda_{\mathrm{TF}}^2/\lambda^2)$ and that the two may usually be used interchangeably, leaving ρ to be calculated from $\nabla\cdot\vec{E}$.

The Schmid-Caroli-Maki[2,3] solution was obtained by assuming that there is a uniform electric field E_0 in the x direction obtained from a scalar potential $\varphi=-E_0x$. Equation (1) was solved to find the moving order parameter Δ, and Eq. (2) was used to obtain the current, which was then averaged. The coefficient relating $\langle j\rangle$ to E_0 was identified as the new conductivity. We find that this procedure does give the correct answer for the average dissipation rate. However, the divergence of their expression for $\vec{j}$ is not zero, and charges must be accumulating to generate a nonuniform E until a steady state is achieved. We will now derive this steady-state solution.

The equation which determines ψ is obtained from the equation of continuity, $\nabla\cdot\vec{j}+\partial\rho/\partial t=0$, and the imaginary part of Eq. (1) multiplied by Δ^*:

$$\gamma\mathrm{Im}[\Delta^*\partial\Delta/\partial t+i2e\psi|\Delta|^2]+\nabla\cdot\mathrm{Re}[\Delta^*(\nabla/i-2e\vec{A})\Delta]=0. \tag{7}$$

The second term is of the same form as the divergence of the second contribution to $\vec{j}$ in Eq. (2). Consequently we arrive at a result, which [noting Eq. (3)] is equivalent to Gor'kov and Éliashberg's[4] Eq. (17),

$$\sigma[\nabla^2\psi+\partial(\nabla\cdot\vec{A})/\partial t]=\gamma[\mathrm{Im}(\Delta^*\partial\Delta/\partial t)(2e)^{-1}+\psi|\Delta|^2](4\pi\lambda^2)^{-1}+\partial\rho/\partial t. \tag{8}$$

This equation introduces a new characteristic screening length ζ for electric fields into the problem, $\zeta^2\equiv4\pi\lambda^2\sigma/\gamma$. For a superconductor with a high concentration of paramagnetic impurities, $\zeta^2=\xi^2/12$. Already one can anticipate that the only possibility for $E(r)$ to be proportional to $B(r)$ is if the two screening lengths are equal, $\zeta=\lambda$.

Near H_{c2} we solve Eq. (8) only to first order in v and $|\Delta|^2$. Unlike Schmid[2] and Caroli and Maki[3] we do not assume that $\vec{E}$ and $\vec{v}$ are in any particular direction relative to the vortex lattice in order to see if there is an anisotropy. The $\partial\rho/\partial t$ term is discarded, as it must be of order v^2, since ρ vanishes (canceled by the lattice charge) in equilibrium.[7] $\partial\vec{A}/\partial t$ is found from the static solution by assuming a uniform translation, $\partial\vec{A}/\partial t = -(\vec{v}\cdot\nabla)\vec{A}$. The first term on the right-hand side of Eq. (8) is similarly obtained from the static solution Eq. (4). Using the fact that the lowering operator of a harmonic oscillator annihilates the Abrikosov solution, $[\partial/\partial x + i(\partial/\partial y - i2eB_0 x)]\Delta = 0$, we can rewrite this term as $-\vec{v}\cdot\mathrm{Im}(\Delta^*\nabla\Delta) = \lambda^2\nabla\cdot(\vec{v}\times\vec{B}) - \vec{v}\cdot\vec{A}_0|\Delta|^2$. Substituting these expressions into Eq. (8) and using a vector identity, we obtain

$$\zeta^2\nabla^2(\psi - \vec{v}\cdot\vec{A}) = (\lambda^2 - \zeta^2)\nabla\cdot(\vec{v}\times\vec{B}) + (\psi - \vec{v}\cdot\vec{A}_0)|\Delta|^2. \tag{9}$$

In terms of the Green's function of the two-dimensional Laplacian operation, $(2\pi)^{-1}\ln|\vec{r} - \vec{r}'|$, we can solve Eq. (9) to first order in $|\Delta|^2$:

$$\psi = \vec{v}\cdot\vec{A} + (\lambda^2/\zeta^2 - 1)\int \ln|\vec{r} - \vec{r}'|\{\nabla'\cdot[\vec{v}\times\vec{B}(\vec{r}')]\} d^2r'/2\pi. \tag{10}$$

Remembering the smallness of λ_{TF}, the electric field may be calculated from $\vec{E} = -\nabla\psi + (\vec{v}\cdot\nabla)\vec{A}$. After using vector identities and integrating by parts we finally obtain

$$\vec{E} = -\vec{v}\times\vec{B} + \left(\frac{\lambda^2}{\zeta^2} - 1\right)\left[-\vec{v}\times\delta\vec{B} + \nabla\times\int\frac{\vec{v}\cdot(\vec{r} - \vec{r}')}{2\pi(\vec{r} - \vec{r}')^2}\,\delta\vec{B}(\vec{r}')d^2r'\right]. \tag{11}$$

This expression verifies our earlier remarks that $\vec{E} = -\vec{v}\times\vec{B}$ only if $\lambda = \zeta$, but that $\vec{E}_0 \equiv \langle\vec{E}\rangle = -\vec{v}\times\vec{B}_0$ nevertheless, since the average of the extra term vanishes.[8]

The charge density is obtained from the divergence of Eq. (11), $\rho = \lambda^2\vec{v}\cdot\nabla\times\vec{B}/4\pi\zeta^2$. The result differs from what would be obtained from the rigid low-velocity Lorentz transformation of the static current distribution $\vec{j}_s = \nabla\times\vec{B}/4\pi$ by the factor λ^2/ζ^2.

Using Eq. (11) we obtain the first contribution to $\vec{j}$ in Eq. (2). To obtain the second contribution it is necessary to solve Eq. (1) for the moving order parameter. This solution is obtained immediately by slightly generalizing that of Schmid, Caroli, and Maki to allow a component v_x:

$$\Delta = \sum_n C_n \exp[-eB_0(x - v_x t - nk/2eB_0 + i\gamma v_y/4eB_0)^2 + i(nk + \tfrac{1}{2}\gamma v_x)(y - v_y t)]. \tag{12}$$

The contribution to the current obtained by substituting this expression into Eq. (2) is the uniformly translating static vortex current distribution j_s (the static B and A are also uniformly translating as assumed earlier) plus an additional term

$$-\gamma(\vec{v}\times\hat{e}_z)|\Delta|^2/16\pi e\lambda^2 = \sigma(-\vec{v}\times\vec{B}_0\xi^2\tfrac{1}{2}\langle|\Delta|^2\rangle + \lambda^2\vec{v}\times\delta\vec{B})/\zeta^2.$$

This term conbined with the others gives the final divergenceless expression for the current,

$$\vec{j} = -\sigma'\vec{v}\times\vec{B}_0 + \vec{j}_s + \sigma\left(\frac{\lambda^2}{\zeta^2} - 1\right)\nabla\times\int\frac{\vec{v}\cdot(\vec{r} - \vec{r}')}{2\pi(\vec{r} - \vec{r}')^2}\,\delta\vec{B}(\vec{r}')\,d^2r', \tag{13}$$

where $\sigma' = \sigma(1 + \xi^2\langle|\Delta|^2\rangle/2\zeta^2)$.

The current expression thus derived is indeed a uniform transport current j_t, the translating static current distribution j_s plus a backflow current j_b, which is necessary when the two screening lengths ζ and λ are unequal. For the model where $\zeta^2 = \xi^2/12$, the ratio $\lambda/\zeta > 1$ when $\kappa = \lambda/\xi > 1/\sqrt{12}$. For these κ values, j_b is antiparallel to j_t at the vortex core, and the current flowing through the vortex core is less than the average transport current j_t. The flow pattern of j_b is that of two small vortices of opposite circulation centered on either side of the main vortex center in the $+\vec{v}$ and $-\vec{v}$ directions.

The extra magnetic field δB_b generated by j_b is obtained using $\nabla\times\delta\vec{B}_b = 4\pi\vec{j}_b$ by simply removing the curl operator in the $\vec{j}_b$ term in Eq. (13). δB_b vanishes at the vortex core and along the line in the $\vec{j}_t$ direction as far as $\delta\vec{B}(\vec{r})$ is symmetric. It initially increases in the $+\vec{v}$ direction and decreases in the $-\vec{v}$ direction when $\lambda > \zeta$. $\langle\delta\vec{B}_b\rangle = 0$, as is necessary not to disturb the flux quantization. In order to preserve the two-dimensional character of the solution we ignore the extra magnetic field generated by j_t. The extra screening currents which arise tend to concentrate j near the surface when the sample thickness $\gtrsim \lambda$ but do not change the average dissipation rate to lowest order. Neither, we now see, does j_b.

At this point it is still not obvious that the average dissipation rate $\langle W\rangle$ is now given by $\langle W\rangle = \sigma' E_0{}^2$. Rather it is necessary to derive an expression for W and calculate the new apparent conductivity. Such an expression has been obtained by Schmid[2]:

$$W=\sigma(\nabla\psi+\partial\vec{A}/\partial t)^2+\gamma|(\partial/2e\,\partial t-i\psi)\Delta|^2(4\pi\lambda^2)^{-1}. \tag{14}$$

In evaluating W, the first term simply gives $\sigma E_0^{\,2}$ to order $|\Delta|^2$ since $\langle\delta E\rangle\equiv 0$ and $\langle(\delta E)^2\rangle$ is of order $|\Delta|^4$. The second term is readily evaluated using Eq. (12) and $\psi_0=\vec{v}\cdot\vec{A}_0$, which gives

$$\gamma B_0 v^2\langle|\Delta|^2\rangle/16\pi e\lambda^2=\sigma E_0^{\,2}\xi^2\langle|\Delta|^2\rangle/2\zeta^2.$$

The initial slope,

$$-(H_{c2}/\sigma)[d\sigma'/dB_0]H_{c2}=\xi^2\kappa^2/\zeta^2[1.16(2\kappa^2-1)+1],$$

which for large κ equals $6/1.16=5.2$, is 3 times larger where the large pair breaking is due to paramagnetic impurities than the result obtained when only the magnetic field is large.[9] Also $\langle W\rangle$ is independent of the direction of $\vec{E}_0$ relative to the vortex lattice to order v and $|\Delta|^2$. Only the backflow, which does not change $\langle W\rangle$, is orientation dependent. The possibility remains that an anisotropy may occur for $\langle W\rangle$ in the higher orders of v or $|\Delta|^2$, which would give a preferred direction for the lattice to line up relative to E_0.

Our solution for the linear dynamic response of the vortex lattice near H_{c2} is now complete. Conservation of charge has led us to introduce an additional new characteristic length ζ necessary to describe dynamic behavior of superconductors. Generally ζ may differ from the screening length λ for magnetic fields. Only if $\lambda=\zeta$ is the local charge density and electric field of the moving vortices given by a rigid Lorentz transformation of the static supercurrent and magnetic field. When $\lambda\neq\zeta$ a qualitatively new feature must arise in order for a steady state to be achieved. This new feature is a backflow current j_b which goes through the vortex cores and returns around their sides. Although j_b arises as part of the contribution to a form resembling a normal current σE, it does not contribute to the dissipation and thus has more the character of a supercurrent. That this lossless current flows directly through the core shows that thinking of the vortex core as a normal region of dimension $\sim\xi$, as in Ref. 1, may be misleading. The order parameter only vanishes at a single point in the core, and the presence of low-lying excitations and absence of a gap do not preclude supercurrents. We have also found similar features in fields well below H_{c2}, which will be presented separately.

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