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Strong Electromagnetic Waves in Overdense Plasmas*

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We report two new results concerning the propagation of electromagnetic waves with a strength parameter $\nu=eE/m\omega c$ sufficiently large ($\nu \gg 1$) to cause relativistic electron velocities. The first is an analytic solution of the nonlinear equations for linearly polarized waves in a uniform medium. The second is that propagation in a nonuniform medium increases the nonlinear penetration effect; the maximum nonrelativistic plasma frequency ω_p which allows transmission of a strong wave is $\omega_p^2 \approx (\omega L/c)^{1/2} eE_i \omega/mc$, where L is the density-gradient scale length and E_i the electric field in the absence of a plasma.

Strong electromagnetic waves, intense enough to make electrons relativistic,[1,2] behave in plasma quite differently from waves of smaller amplitude. This Letter deals with propagation of waves with a strength parameter $\nu=eE/m\omega c \gg 1$ in regions where the plasma density varies on scale lengths large compared to the vacuum wavelength. Previous work[3] on the acceleration of particles by strong waves has largely concentrated on regions roughly a wavelength in size.

Solutions of the nonlinear equations governing the propagation of circularly polarized waves in uniform plasmas are available[2] and show that strong waves may propagate in overdense plasmas provided the peak electric field E satisfies the inequality

$$1-\frac{\omega_p^2}{\omega_p}\left[1+\left(\frac{eE}{mc\omega}\right)^2\right]^{-1/2} \approx 1-\frac{\omega_p^2}{\omega}\left(\frac{mc}{eE}\right)>0. \quad (1)$$

In this Letter we derive an analytic solution for the linearly polarized wave, previously studied by single-particle[3] or numerical[2,4] methods, in the regime where self-consistent collective effects are important and we show that it possesses a propagation condition very similar to (1). The particle and field energy fluxes of these solutions are then combined with a requirement of propagation at constant energy flux and the condition that a WKB description be valid to derive the propagation condition for electromagnetic waves incident on a plasma slab from vacuum. The propagation condition is

$$\omega_p^2 < (\omega L/c)^{1/2} eE_i\,\omega/mc, \quad (2)$$

to within factors of order unity. Here E_i denotes the electric field strength in a vacuum and L is the density-gradient scale length.

The model is a uniform, cold plasma with fixed ions and no magnetic field. Akhiezer and Polovin[1] have derived the nonlinear equations governing linearly polarized waves propagating in the z direction:

$$\frac{d^2\rho_x}{d\zeta^2}+\left(\frac{1}{\beta^2-1}\right)\frac{\beta\rho_x}{\beta(1+\rho^2)^{1/2}-\rho_z}=0, \quad (3)$$

$$\frac{d^2}{d\zeta^2}\left[\beta\rho_z-(1+\rho^2)^{1/2}\right]+\frac{\rho_z}{\beta(1+\rho^2)^{1/2}-\rho_z}=0. \quad (4)$$

Here $\vec{\rho}=\vec{\mathrm{p}}/mc$ is the dimensionless electron momentum, βc is the phase velocity, and all spatial and temporal dependence occurs in the combination $\zeta=\omega_p c^{-1}(z-\beta ct)$, where ω_p is the nonrelativistic plasma frequency. The formulas for the electric field, etc., in terms of $\vec{\rho}$ are in the paper of Akhiezer and Polovin.[1]

We consider large-amplitude waves ($|\rho_x|_{\max} \gg 1$) in regimes where collective effects are important so that $\beta \gg 1$. In these circumstances, periodic solutions have $|\rho_x| \gg |\rho_z|$ for most of each period. Equation (3) then indicates that ρ_x

is a series of linked parabolas:

$$\begin{aligned}\rho_x &= \rho_0 - \frac{(\zeta - \frac{1}{4}P)^2}{2(\beta^2-1)}, \quad 0 \leq \zeta \leq \tfrac{1}{2}P, \\ &= -\rho_0 + \frac{(\zeta - \frac{3}{4}P)^2}{2(\beta^2-1)}, \quad \tfrac{1}{2}P \leq \zeta \leq P,\end{aligned} \tag{5}$$

where $\rho_0 = (\frac{1}{4}P)^2[2(\beta^2-1)]^{-1} \gg 1$. The corresponding solution for ρ_z is obtained by rescaling (4):

$$\frac{d^2}{d\eta^2}\left\{R - \left[(1-\eta^2)^2 + \frac{R^2}{\beta^2} + \frac{1}{\rho_0^2}\right]^{1/2}\right\} + \frac{2R}{[(1-\eta^2)^2 + R^2/\beta^2 + 1/\rho_0^2]^{1/2}} = 0, \tag{6}$$

where $\eta \equiv 4\zeta/P - 1$ and $R \equiv \beta\rho_z/\rho_0 = O(1)$. The doubly periodic nature of R requires that $dR/d\eta = 0$ at $\eta = 0$ and $\eta = 1$. Equation (6) is solved separately in two regions: a boundary layer near $\eta = 1$ and $0 \leq \eta < 1$. The two solutions are matched for $1 \gg 1 - \eta \gg (\beta^{-2} + \rho_0^{-2})^{1/2}$, where their regions of validity overlap. Away from the boundary layer, η satisfies the inequality $1 - \eta \gg (\beta^{-2} + \rho_0^{-2})^{1/2}$ and the small terms of order β^{-2}, ρ_0^{-2} in the square brackets can be neglected. The solution is[5]

$$R = A(1-\eta^2) - \tfrac{2}{3}\eta^2 + \tfrac{1}{3}(1-\eta^2)\ln(1-\eta^2), \tag{7}$$

where A is arbitrary. When $1 \gg 1 - \eta \geq 0$, the solution valid as $\eta \to 1$ has R almost constant and the asymptotic expansion

$$R(\eta) = R_0 + (2 + R_0)(1-\eta) - \tfrac{1}{2}R_0(1-\eta)\ln\left(\frac{16(1-\eta)^2}{R_0^2\beta^{-2} + \rho_0^{-2}}\right), \tag{8}$$

where $R_0 = R(1)$. Matching (7) and (8) in their region of common validity yields

$$R_0 = -\tfrac{2}{3}, \quad A = \tfrac{1}{6}\ln\left(\frac{36\beta^2\rho_0^2}{4\rho_0^2 + 9\beta^2}\right). \tag{9}$$

Figure 1 illustrates the linearly polarized solution.

What is the propagation condition for these linearly polarized waves? Combining the formula for ρ_0 [following (5)] with the relation[1] $e^2E^2(m^2\omega_p^2c^2)^{-1} = 2\beta^2\rho_0(\beta^2-1)^{-1}$ yields the condition

$$\frac{1}{\beta^2} = \frac{c^2k^2}{\omega^2} = 1 - \frac{\pi}{2}\frac{\omega_p^2}{\omega}\frac{mc}{eE} > 0, \tag{10}$$

almost identical to (1).

Now we turn to the propagation of strong waves in nonuniform plasmas. The model is that of a strong wave normally incident on a plasma slab with fixed ions. Our goal is to derive conditions under which the wave will be transmitted through the slab without reflection. When this is the case, the solution continues to be a function of t only through the phase ζ, although the amplitude is a slowly varying function of z. In the absence of a reflected wave, the energy flux of the incoming wave is constant. The energy flux S associated with a circularly polarized wave is purely electromagnetic,

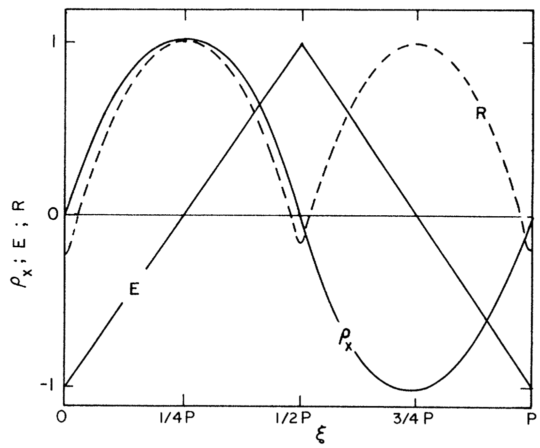

FIG. 1. Strong linearly polarized electromagnetic waves. The curves show transverse ρ_x and longitudinal R components of electron momentum [see Eq. (6)] as well as the transverse electric field. All quantities have been normalized to their maximum value. The period is P.

$$S = cE^2/4\pi\beta, \tag{11}$$

while linearly polarized wave has contributions from both particles and fields,

$$S = \frac{cE^2}{12\pi\beta}[1 + \tfrac{3}{2}\langle R\rangle] = \frac{cE^2}{72\pi\beta}\left[\ln\left(\frac{576\beta^2\rho_0^2}{4\rho_0^2 + 9\beta^2}\right) + \frac{2}{3}\right], \tag{12}$$

with β given by (10) and $\langle R\rangle$ denoting the average of R over one period. The combination of (11) and (12) with (1) and (10) predicts that there is no point where $\beta \to \infty$ for finite plasma densities and constant energy flux. Therefore, the wave continues to be transmitted through the slab. This is a consequence of nonlinear effects: As the incident wave penetrates to higher density, the electric fields increase by (11) faster than the first power of ω_p^2, allowing further penetration of the plasma according to (10).

However, the wave is not transmitted for arbitrarily high densities. The WKB concepts on which (11) and (12) are based will be violated if there exists a point where the wavelength exceeds the scale length over which the phase velocity changes. For small-amplitude wave propagation, this occurrence is identified with the reflection of the wave. In the present nonlinear problem it is difficult to make the analogous identification, since the presence of a reflected wave alters the solution for the incoming wave as well. To circumvent this difficulty, we seek a regime where it is consistent to assume that there is propagation with no reflected wave present, i.e., where the wavelength of the incident wave is less than the scale length for phase-velocity variation. These quantities can readily be estimated for the circularly polarized wave. (The linearly polarized wave is not qualitatively different, but the logarithms complicate the algebra.) Since $\beta \gg 1$, (10) gives

$$e^2E^2/m^2c^2 \approx \omega_p^4/\omega^2;$$

and solving (11) for β, one finds

$$\beta \approx \nu_i^{-2}\omega_p^4\omega^{-4},$$

where $\nu_i = eE_i/mc\omega$ gives the incident electric field strength in the vacuum. Therefore $\beta'/\beta = 2L^{-1}$, L being the density-gradient scale length. Transmission will occur as long as

$$\tfrac{1}{2}kL = \tfrac{1}{2}(\omega/c\beta)L = \tfrac{1}{2}\omega L\omega^4\nu_i^2/c\omega_p^4 > 1. \qquad (13)$$

Solving (13) for ω_p^2 yields the propagation condition (2).

The reason why strong electromagnetic waves penetrate plasmas more readily than small-amplitude waves is that the plasma current is limited to the value nec, instead of increasing with E as $ne^2E/m\omega$. A wave will be reflected only if the plasma current is large enough to cancel the displacement current. Relativistic effects thus diminish the ability of the plasma to act as a dielectric. The model of fixed ions limits our calculations to values of $\nu = eE/mc\omega$ in the range $1 < \nu < M_i/m_e$. Qualitatively one expects that when ions become relativistic, the current they generate will be subject to the same limitation and will not cause major changes in the propagation condition. In linearly polarized waves with phase velocity large compared to the velocity of light, both the longitudinal motion of electrons and their density perturbations are small because of the self-consistent electrostatic fields. Nevertheless, the longitudinal motion is important since particle energy flux exceeds field energy flux in linearly polarized waves. One additional caution must be given: The solutions presented here may not be stable; indeed strong ac fields lead to instabilities in nonrelativistic plasmas.[6]

Our work demonstrates that sufficiently strong linearly polarized waves can propagate in plasmas where $\omega_p^2 > \omega^2$ although a recent paper by Noerdlinger[3] argued against this possibility. Noerdlinger neglected electrostatic fields and did not clearly specify which regions of retarded time contribute to radiation by current sheets. We have considered the problem of an electromagnetic pulse incident on a plasma density step. The results show a simultaneous occurrence of two effects: First, the longitudinal electrostatic field becomes large enough to balance the $\vec{v}\times\vec{B}$ acceleration. Second, the magnetic fields generated by the plasma become comparable to the wave magnetic fields. These effects combine to limit the longitudinal electron velocity below the velocity used by Noerdlinger and thus alter his propagation condition. A complete description of how our periodic solutions develop inside a plasma from a wave initially incident on the plasma's surface awaits further work.

The strong electromagnetic radiation believed to occur in pulsars[3,7] is the principal astrophysical application of our results. Figure 2 shows the maximum permissible plasma density for transmission of the radiation according to (2), with the additional assumption that the plasma-density scale length L is of the order of the distance R from the pulsar. The term involving L in (2) is an important feature not found in previ-

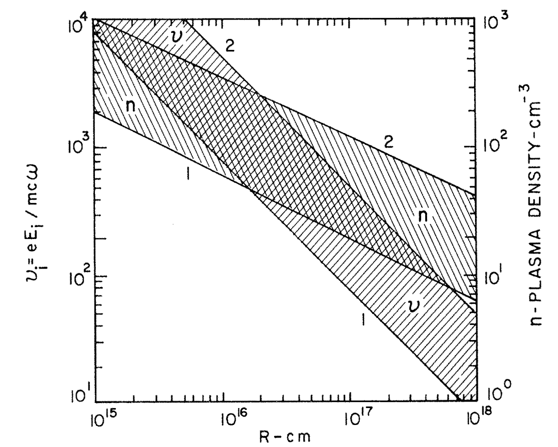

FIG. 2. Vacuum strength parameter $\nu_i = eE_i/mc\omega$, and maximum plasma density [see Eq. (2)] for transmission of electromagnetic waves, versus radius R. The curves are based on magnetic dipole radiation from the Crab pulsar (cf. Refs. 3 and 7) with surface fields in the range from (1) 6×10^{11} to (2) 4×10^{12} G. In evaluating (2), we assume that $L = R$.

ous theories.[2,3,7] One can conclude that it may be possible for pulsar radiation to penetrate to large distances in the medium of the Crab nebula, thus allowing the possibility of an energy source in these large volumes. On the other hand, penetration of the filaments with their higher densities and shorter scale lengths appears doubtful. Thus, the results presented here would support the hypothesis that the "amorphous-mass" region of the Crab nebula is filled with strong electromagnetic radiation, since the electron density in this region is less than 1 cm^{-3}.[8]

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Mobility of Electrons on the Surface of Liquid ^{4}He

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The mobility of electrons on the surface of liquid ^{4}He has been measured from 0.9 to 3.2°K. Although scattering from atoms in the vapor appears to be the dominant scattering mechanism, the mobility of the surface electrons is significantly lower than that of free electrons in the vapor. No evidence for surface-wave scattering is discernible.

The existence of extrinsic image-potential–induced surface electron states in certain insulators, including liquid helium, has been discussed by Cole and Cohen.[1] Subsequently, several mechanisms which might influence the properties of electrons in surface states on liquid helium have been proposed. Cole[2] has predicted that scattering by quantized surface waves would be the dominant scattering mechanism at low temperatures and would lead to an essentially temperature-independent mobility below about 1°K. More recently Crandall and Williams[3] have discussed the possibility of crystallization of the surface electrons.

To investigate the interactions of electrons in surface states on liquid ^{4}He, we have measured the mobility of surface charges between 0.9 and 3.2°K using the apparatus shown schematically in Fig. 1(a). A movable support positions the chamber containing the electrodes so that the liquid surface is at a height h of 1 to 2 mm above the set of three identical submerged electrodes. Electrons are provided by a gas discharge located in the vapor about 8 cm above electrode 2. The potential difference V_{dc} between electrode 2 and the submerged electrodes, along with the distance of these electrodes below the surface, determines the surface charge density. A signal from the internal oscillator of a lock-in amplifier is applied to an outer electrode of the submerged set. The signal coupled by the surface charge to the other outer electrode is amplified by the lock-in amplifier and compared in phase with the input signal, yielding data from which the surface electron mobility may be determined. Direct capacitive coupling between the two outer electrodes is reduced to a negligible value by the grounded central electrode.

In normal operation the electron source was operated continuously. When accumulated charge reduced the field between electrode 2 and the surface to zero, the surface would accept no more charge and the entire potential difference V_{dc} would appear between the surface and the submerged electrodes. If C represents the capacitance per unit length between the surface and the submerged electrodes then it will be seen that in