Nonconservation of the Newman-Penrose Conserved Quantities*

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We have examined the Newman-Penrose quantities for test fields in a Schwarzschild background. We find that, unless a static condition has prevailed in the infinite past, backscattered electromagnetic and gravitational waves make it impossible to define the quantities as limits at future null infinity. An operational definition in terms of observations at finite radii is possible, but yields quantities which are not conserved.

An observer located at some distance from a bounded source measures its field. If the source is an electric charge-current distribution, he measures the electromagnetic field $F_{\mu\nu}$. If it is an energy-momentum-stress distribution, he measures the gravitational Riemann tensor $R_{\nu\rho\sigma}{}^{\mu}$. Surround the source region by a number of observers, each carrying a local coordinate frame whose orientation is known, and combine their measurements to extract global information about the field. For example, by performing an angular integral calculate the monopole strength of the electromagnetic field (radial electric field integrated over the sphere). We may imagine the considerable surprise and interest accompanying the observational discovery that the number obtained is a constant in time—even when the fields measured by each local observer vary in time.

Likewise, there has been considerable interest in the apparently conserved quantities, discovered by Newman and Penrose, associated with electromagnetic and gravitational fields in asymptotically flat space-times.[1] There are six electromagnetic Newman-Penrose quantities (NPQ's) and ten gravitational NPQ's. The Maxwell and Einstein equations appear to guarantee that these quantities will be constant in time. However, their physical significance has been obscure, and no attempt to relate them to conserved properties of the sources (as a time-invariant electric monopole measures the electric charge) has succeeded.[2] It is difficult to study the properties of the NPQ's in detail because they are trivial for electromagnetic fields in flat space and for gravitational fields linearized about flat space. (They vanish if radiation is purely outgoing.)

Recent developments[3] in the theory of perturbations of the Schwarzschild geometry have enabled us to analyze the NPQ's in this nontrivial case. We find that, unless the source was static in the infinite past, backscattered electromagnetic and gravitational waves make it impossible to define conserved NPQ's as limits at future null infinity. An operational definition of NPQ's is possible in terms of observations at finite radii. But the resultant NPQ's are not conserved; changes in them propagate outward with about $\frac{1}{3}$ the speed of light from epochs when the source is nonstatic.

In this Letter we use the electromagnetic NPQ's as an example. Full mathematical details for both electromagnetic and gravitational perturbations of the Schwarzschild metric will appear elsewhere.

The six components of the electric and magnetic fields can be combined into three complex scalar quantities by projection of the field tensor onto a complex, null tetrad. In the notation of Price,[3] these are

$$\phi_{+1} = F_{\mu\nu} l^{\mu} m^{\nu}, \tag{1a}$$

$$\phi_0 = \tfrac{1}{2} F_{\mu\nu} (l^{\mu} n^{\nu} - m^{\mu} m^{*\nu}), \tag{1b}$$

$$\phi_{-1} = F_{\mu\nu} m^{*\mu} n^{\nu}. \tag{1c}$$

It is ϕ_{-1} that contains the dominant outgoing radiation field. However, each scalar contains complete information about the entire electromagnetic field, since the other two can be computed from it with the help of the Maxwell equations.

While the NPQ's are usually computed from ϕ_{+1}, they can just as well be computed from ϕ_0 (or ϕ_{-1}). The equation governing ϕ_0 is the closest to a conventional wave equation, and this simplifies the physical interpretation. To identify the NPQ's in ϕ_0, one needs the results of two successive organizations of measurements by observers surrounding the source. First, for each r, t select the *dipole* part of ϕ_0 by an angular integral over an $l=1$ spherical harmonic:

$$\phi_{\text{dipole}}(t, r) = \int Y_{1m}(\theta, \varphi) \phi_0(t, r, \theta, \varphi) \times \sin\theta \, d\theta \, d\varphi. \tag{2}$$

The analogous *monopole* integral would yield directly the conserved charge. To obtain the NPQ's one must analyze the detailed radial dependence of $\phi_{\text{dipole}}(t, r)$ along an outgoing radial null line u

= const. In the Schwarzschild background,

$$u \equiv t - r^* \equiv t - r - 2M \ln(r/2M - 1). \qquad (3)$$

The conventional expansion for outgoing waves assumes that at sufficiently large r the radial dependence can be analyzed as a power series in r^{-1},

$$\Psi(u,r) \equiv r^2 \phi_{\text{dipole}}(t,r) = f_0(u) + f_1(u)/r + f_2(u)/r^2 + \cdots. \qquad (4)$$

The NPQ is (except for a factor 2) the coefficient $f_2(u)$, which is actually three complex quantities corresponding to the three values for m in Eq. (2).

The Maxwell equations applied to the form (4) yield the result that $f_2(u)$ is independent of u. Thus, the NPQ is apparently conserved. Specifically, Ψ satisfies

$$\Psi_{,tt} - \Psi_{,r^*r^*} + (2/r^2)(1 - 2M/r)\Psi = 0. \qquad (5)$$

This, after a change of variables to u and r, yields equations relating the f_n's:

$$f_1' = f_0, \qquad (6a)$$

$$f_2' = 0, \qquad (6b)$$

$$\cdots$$

$$f_n' = -\frac{(n+1)(n-2)}{2n} f_{n-1} + (n-2)M f_{n-2}. \qquad (6c)$$

The function $f_1(u)$ is arbitrary; it is the relativistic generalization of the flat-space dipole moment of the sources. It is determined, in principle, by matching to an interior solution in the source region. In flat space only f_1 and f_0 would be nonzero. The functions $f_n(u)$, $n \geq 3$, are obtained by successive iterations of Eq. (6c) and represent relativistic corrections to the flat-space propagation of the fields; i.e., they represent backscatter.[4] Equations (6c) also determine the *static* values of f_2 and the f_n in terms of f_1. In particular,

$$f_2 = \tfrac{3}{2}M(f_1)_{\text{static}}. \qquad (7)$$

The paradox of the NPQ's is that, for an initially static source, the value of f_2 is determined by the initial static value of f_1, and it cannot change even after f_1 and the f_n become nonstatic. However, it seems physically necessary that a motion of the sources which lasts for a finite retarded time should generate a radiation field that becomes static, at least asymptotically, in the future. A net change in f_1 due to the motion of the sources should then produce a new static value of f_2, given by Eq. (7)—which it cannot, because Eq. (6b) demands that f_2 be constant.

The resolution of the paradox can be illustrated by a simple example. Assume that, at retarded time $u=0$, $f_1(u)$ changes instantaneously from an initial static value D to a new constant value D'. Direct integration of Eq. (6) gives, $u>0$,

$$\Psi = \frac{D'}{r} + \frac{\frac{3}{2}MD}{r^2} + \sum_{n=3}^{\infty} \frac{2M(D'-D)(-)^{n+1}(n+1)u^{n-2}}{(2r)^n} + O\left(\frac{M^2}{r^3}, \frac{M^2 u}{r^4}, \cdots\right). \qquad (8)$$

The terms neglected are small compared to those kept for all u/r, as long as $r \gg M$. Evidently, as long as the series converges, the NPQ ($\frac{3}{2}MD$) "remembers" the old value D and is conserved. But the series diverges for $u > 2r$—i.e., it diverges inside a sphere that moves outward at asymptotically $\frac{1}{3}$ the speed of light.[5] When the series diverges, the NPQ's become ill defined and cannot be evaluated uniquely from data in that region. Only if observers extend *all* the way to $r = \infty$ can the conserved NPQ be evaluated for all u: *The quantity measures the old static dipole moment using increasingly distant data, and is in no way related to the motion of the sources after they become nonstatic.*

Does space-time at fixed r become asymptotically static again? Analytically continue the solution for Ψ to $u > 2r$ by summing the dominant terms in (8). The result is

$$\Psi = \frac{D'}{r} + \frac{3}{2}\frac{MD}{r^2} + \frac{3}{2}\frac{M(D'-D)}{r^2}\frac{u(u+8r/3)}{(u+2r)^2} + O\left(\frac{M^2}{r^3}\right). \qquad (9)$$

As $u \to \infty$ with r fixed,

$$\Psi \to \frac{D'}{r} + \frac{\frac{3}{2}MD'}{r^2} + O\left(\frac{M}{ur}, \frac{M^2}{r^3}\right). \qquad (10)$$

When $u \gg 2r$, local measurements will yield an apparent *new* value of the NPQ which is related to the new value of the static dipole moment in the same way the old value of the NPQ was related to the old value of the static dipole moment. In a strict mathematical sense this value is not the "correct" NPQ obtainable from data inside the cone $u = 2r$. Operationally, with finite measurement errors, there is no way to distinguish an "apparent" value from a "correct" one; in a real, physical sense the NPQ has changed its value.

Thus the paradox is resolved: A mathematically precise NPQ is conserved if it exists, but it only exists outside the $\frac{1}{3}$-speed-of-light cone from when the source *first* becomes nonstatic. An operationally measurable NPQ may be defined each time the source becomes static again for a sufficiently long interval, but it can change in value between the different static epochs.

The above discussion lacks mathematical rigor. However, the same conclusions can be derived rigorously from an expansion of the form

$$\Psi = f_0(u) + \frac{f_1(u)}{r} + \frac{M}{r^2}g_1(r,u) + \frac{M^2}{r^3}g_2(r,u) + \cdots, \tag{11}$$

which is valid for all u at all $r \gg M$. Summation of the leading terms in (8) is equivalent to a calculation of $g_1(r,u)$, the first-order backscatter.

Why do the NPQ's fail at $u = 2r$? An analytic solution to the *vacuum*-field equations with an outgoing boundary condition must become singular on some *past* light cone of Schwarzchild space-time. This is because the generic outgoing-wave solution is generated by a source, so if we propagate a solution backward in time refusing to insert a source, we encounter a singularity on the locus of our "last chance" to put the source in; this, essentially, is a past light cone. Now recall that a power series converges or diverges in a disk in the complex plane. For $r \gg M$, the dominant contribution to Ψ is essentially a power series in u/r [e.g., Eq. (8)]. Hence a physical singularity at $u = -2r$ leads to a divergence of the series at $u = +2r$. In short, the $\frac{1}{3}$-speed-of-light sphere is a mathematical ghost of the past light cone, a consequence of the prescription that we analyze the data in u, r coordinates to obtain the NPQ's. [It will be noted that the past light cone is actually $u = -2r^*$, not $u = -2r$. This sloppiness arises from our neglect of terms of order M^2 in Eq. (11).]

There are mathematical difficulties with the NPQ's even when they are defined at future null infinity. A net change of the electromagnetic dipole moment in a burst of radiation of $u \simeq u_0$ generates a line $u - u_0 = 2r$ across which the NPQ's cannot be continued as conserved quantities. These lines, from a series of such changes at progressively earlier times, are everywhere timelike, but they accumulate at future null infinity (see Fig. 1). If the amplitude of the changes in the dipole moment does not go to zero (i.e., if the dipole moment is not asymptotically static in the infinite past), the limit defining the NPQ's

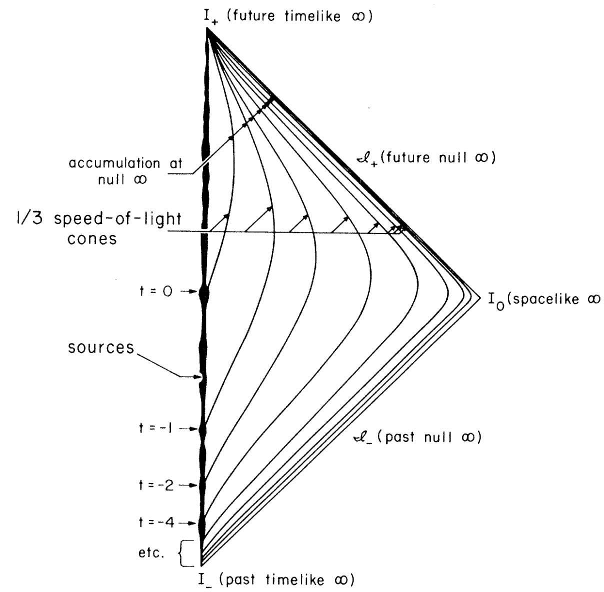

FIG. 1. By a conformal transformation, the infinite ranges of r and t are shown in a Penrose-Carter diagram. The Newman-Penrose quantities in Schwarzschild space-time cannot remain conserved across lines which originate when the field sources are nonstatic, and which travel outward at approximately $\frac{1}{3}$ the speed of light. If the sources have been nonstatic for all time, there is an accumulation of these lines at null infinity—even though each line is everywhere timelike. Shown here are the "$\frac{1}{3}$-speed-of-light cones" originating at times $t = 0, -1, -2, -4, \cdots, -64$; the accumulation is already evident. If follows that the Newman-Penrose quantities cannot be defined as a limit at null infinity.

at future null infinity does not exist. This is generally the case even if the sources have radiated a finite energy in their infinite past history.

We conclude that the NPQ's in a Schwarzschild background represent an information structure in the curved-space propagation of waves. The NPQ's "remember" an initial static value of the dipole moment (electromagnetism) or the quadrupole moment (gravitation), but for an observer at fixed r they remember it only for a finite time. If defined at null infinity, the NPQ's exist only for a source which was asymptotically static in the infinite past. If measured at finite r, they can be defined (or redefined with a *new* value) when the source has been static for a long time, $\Delta u \gg r$.

The conclusions outlined here are equally valid for electromagnetic NPQ's in any metric theory of gravity (e.g., Brans-Dicke), for gravitational NPQ's in general relativity, and for NPQ's associated with nongravitational fields of any integer spin.

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Matter in Superstrong Magnetic Fields: The Surface of a Neutron Star*

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In huge magnetic fields ($B \gtrsim 10^{12}$ G) matter forms a tightly bound, dense ($\gtrsim 10^4$ g cm^{-3}) solid with properties of a one-dimensional metal and a work function of the order of a keV. Electron field emission from the sharp surface of a pulsar is much easier than ion emission; it is estimated to be cut off when the stellar rotation period exceeds several seconds.

The enormous magnetic fields ($B \sim 10^{12}-10^{13}$ G) which are assumed to thread the surface of canonical neutron stars (pulsars) dominate the motion of electrons and the structure of matter in the stellar surface. The form of such matter and its properties are discussed below, together with a possible consequence for pulsar observations.

In minimum-energy degenerate eigenstates for electrons in a uniform magnetic field $\vec{B}$ the particles are effectively confined to move in tubes of radius

$$\hat{\rho} \equiv \frac{\hbar}{mc}\left(\frac{m_e{}^2c^3}{\hbar e B}\right)^{1/2} \sim \frac{2.6\times 10^{-4}}{B^{1/2}} \text{ cm.} \tag{1}$$

around a flux line. The infinite degeneracy associated with the arbitrariness in position of the flux line is most conveniently exploited in problems with cylindrical symmetry by using cylindrical eigenstates of approximate radius

$$\rho_n = (2n+1)^{1/2}\hat{\rho} \quad (n=0,1,2,\cdots) \tag{2}$$

on which the electron motion is centered ("Landau orbitals").[1] All lowest-state electrons have the same spin direction (antiparallel to $\vec{B}$). Excited states which describe spin flip or motion greater than zero point perpendicular to $\vec{B}$ are excited by integer multiples of $e\hbar B(mc)^{-1} \sim 12B_{12}$ keV (where B_{12} is the magnetic field in units of 10^{12} G). The canonical magnetic fields of pulsars are large enough that the excited states do not enter significantly into the description of free atoms or condensed stellar surface matter.

The strength of a magnetic field on an atom of atomic number Z is characterized by the dimensionless parameter

$$\eta \equiv \frac{a_0}{Z\rho_Z} = \left(\frac{B}{4.6\times 10^9 Z^3}\right)^{1/2}, \quad Z \gg 1, \tag{3}$$

which is the ratio of the Bohr radius of the most tightly bound electron when $\vec{B}=0$ to the cylinder radius of the atom formed by putting exactly one electron into each Landau orbital. There are three qualitative regimes[2,3]: (i) $\eta \gg 1$ (ultrastrong B), (ii) $1 \gg \eta \gg Z^{-3/2}$ ("strong" B), and (iii) $Z^{-3/2} \gg \eta$ (perturbative B). In regime (iii) $\vec{B}$ is sufficiently weak that conventional perturbation treatments are adequate. In the ultrastrong field regime (i) the lowest-energy state of a single atom is achieved by successively putting single electrons into Landau orbitals which keep them (in directions perpendicular to $\vec{B}$) much closer to their nuclei than can the nuclear Coulomb field alone. The resulting atoms[2-4] are small and elongated along $\vec{B}$ with energies (relative