controlled conditions.

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Regularization of the Maki Conductivity by Fluctuations*

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The Maki conductivity for a superconductor above T_c is regularized by taking account of fluctuation effects in the impurity vertex correction.

In a microscopic calculation of the contribution of superconducting pair fluctuations to the dc electrical conductivity of a superconductor above T_c, two terms give the critical temperature dependence. The first, computed by Aslamazov and Larkin[1] (AL), is for a thin film

$$\sigma_{\mathrm{AL}}' = e^2/16dt, \tag{1}$$

where d is the film thickness and t the reduced temperature, $t = \ln(T/T_c) \simeq (T - T_c)/T_c$. We have set $k_B = \hbar = 1$. Extensive experiments[2] on highly disordered films of Bi, Pd, and Ga show excellent numerical agreement with this term alone. The second term, calculated by Maki,[3] is infinite at all temperatures for one- and two-dimensional samples. Experimental results on aluminum films[4-6] deviate strongly from Eq. (1) and have been fitted, following Thompson,[7] by including the Maki conductivity, made finite by the assumed presence of a pair-breaking interaction. The Maki-Thompson conductivity is

$$\sigma_{\mathrm{MT}}' = (e^2/8d)\ln(t/\delta)(t-\delta)^{-1}, \tag{2}$$

where δ is the pair-breaking strength. By fitting the sum of Eqs. (1) and (2) to a variety of films, δ for aluminum has been found to have the empirical value $\delta = 5\times10^{-4}R_{\square}$, proportional to the normal resistance per square of the film, measured in ohms. The origin of the pair breaking has been ascribed to proximity effects, paramagnetic

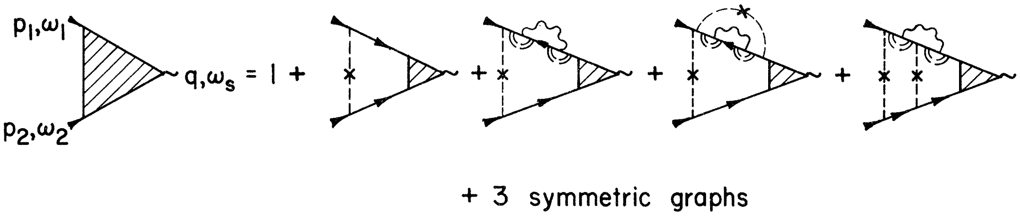

FIG. 1. Integral equation for the pair vertex. Broken lines denote impurity scattering, wavy lines are pair fluctuations, and solid lines are electron propagators whose self-energy contains the impurity scattering, but not fluctuations.

impurities, or electron-phonon interactions.

In practice these effects may play an important role in reaching agreement with experiment. However, it is unsatisfactory to be required to go outside the BCS model to achieve finite conductivity in a dirty system. If this were necessary, it would be the first qualitative failure of the BCS theory.

The standard calculation for dirty systems is an expansion of the conductivity tensor in powers of fluctuations. The divergence of the Maki conductivity is a consequence of the divergence of the impurity vertex correction, associated with two electrons forming a pair, when the electron frequencies and the pair momentum go to zero.

We have removed the divergence within the BCS model by correcting the impurity vertex for the effects of fluctuations. We write an integral equation for the vertex whose kernel contains the important terms which are of first order in fluctuations.[8] The vertex can be split into a part which depends on the incoming electron momenta and one which does not. The equation for the momentum-independent part is shown in Fig. 1. The momentum-dependent part of the vertex is given by the last diagram of Fig. 1, with the outer impurity line removed. Keeping only terms which can become singular when $T\rightarrow T_c$, the new expression for the momentum-independent part of the vertex function is (when $\omega_1\omega_2<0$)

$$\Lambda(\omega_1,\omega_2,q)=(|\omega_1-\omega_2|+\tau^{-1})\{(|\omega_1-\omega_2|+Dq^2)[1+M(|\omega_1|)+M(|\omega_2|)]+L(|\omega_1|)+L(|\omega_2|)\}^{-1}, \tag{3}$$

where

$$L(|\omega_n|)\simeq\frac{8T^2}{N(0)\pi}\sum_q\frac{1}{Dq^2+\epsilon}\,\frac{1}{|2\omega_n|+2Dq^2+\epsilon},$$
$$M(|\omega_n|)\simeq-\frac{1}{2}\frac{\partial L(|\omega_n|)}{\partial|\omega_n|}, \tag{4}$$

$\epsilon=(8T/\pi)t$, τ is the lifetime for impurity scattering, $N(0)$ the electronic density of states, and D the diffusion constant. In the following we ignore M since, when it becomes important, the approximation of keeping only first-order terms in the vertex kernel is already unsatisfactory. In two dimensions we have

$$L(|\omega_n|)=\frac{T^2}{\pi^2N(0)dD}\,\frac{1}{(|\omega_n|-\frac{1}{2}\epsilon)}\ln\left(\frac{|\omega_n|+\frac{1}{2}\epsilon}{\epsilon}\right). \tag{5}$$

We have recomputed the conductivity of a thin film, keeping the usual first-order terms in the expansion but inserting the renormalized vertex function of Eq. (3). Because the momentum-dependent part of the vertex is of higher order we may neglect it. Since in the Maki conductivity the vertex contributes near $\omega_1,\omega_2\rightarrow0$, we may replace L by its value at $\omega=0$. Then the Maki conductivity is again given by Eq. (2), but now δ is strongly temperature dependent and has the value

$$\delta_1=(\pi/4T)L(0)=2.11\times10^{-5}R_\square/t. \tag{6}$$

In the AL term (and in the renormalization of the fluctuation propagator) the vertex only contributes at frequencies $|\omega_n|\geqslant\pi T$. Because of the strong frequency dependence of L shown in Eq. (5), the correction of the vertex is small at these frequencies and may be neglected. Then Eq. (1) is unchanged. It is interesting that the resultant conductivity is independent of all material parameters except the normal resistance per square of the film and the reduced temperature. It is quite satisfactory that the computed value δ_1 is comparable to the measured δ near T_c.

The calculation outlined above is essentially a selective resummation of perturbation theory, keeping terms to all orders only where necessary to remove a divergence. Many terms have been omitted from the conductivity which would be needed to ensure gauge invariance. We believe that the effect of these terms is small. In any case the Meissner current is zero in this approximation. Now in impure systems in the absence of other interactions the pair vertex is closely related to the particle-hole (p-h) vertex whose divergence in the limit of long wavelengths and small electron frequencies is required by gauge invariance. It might be thought that a gauge-invariant calculation of the pair vertex in the presence of interactions must also remain divergent. This is not the case. Fluctuation corrections to the p-h vertex and pair vertex are quite different; the p-h vertex has first-order corrections resembling the Maki and AL diagrams in addition to those of Fig. 1. Furthermore we have constructed a manifestly gauge-invariant model calculation, φ-derivable in the sense of Baym,[9] in which the pair-vertex divergence is removed much more strongly than in the present case, though the p-h vertex remains divergent. In the model calculation the final diagram of Fig. 1 is omitted. A clue as to the nature of interactions which remove the pair-vertex divergence is perhaps provided by the observation that the contributions to L of the final three terms in Fig. 1 cancel exactly unless we take account of the change of momentum and frequency of an electron when interacting with a fluctuation.

At temperatures sufficiently close to T_c, our linear approximation (of keeping only first-order terms) becomes inaccurate, as interactions between fluctuations become important. In the AL conductivity, the fluctuation propagator, and other expressions involving the vertex only at finite frequency, the criterion for accuracy of the linear approximation is $L(\pi T) \lesssim \epsilon$ which may also be written as $t \gtrsim 10^{-4} R_\square$. This is the same as the widely used Ginzburg criterion,[10] derived in connection with the breakdown of a mean-field theory. In the Maki diagram, however, the fluctuations contribute at zero frequency where they are much stronger. The condition now is $M \lesssim 1$ or $L(0) \lesssim \epsilon$ which gives $t \gtrsim \delta_1$, or $t^2 \gtrsim 10^{-5} R_\square$. This criterion implies a greatly enlarged critical region. The failure of the Ginzburg criterion for the Maki conductivity is related to the absence of this term in phenomenological calculations arising from mean-field theory.

Although our major purpose was to demonstrate that the Maki conductivity is finite in the BCS model, we have also found rather good agreement with experiments on aluminum. Figure 2 shows experimental data for an Al film with $R = 3.31\ \Omega$. Curve a is the sum of Eqs. (1) and (2)

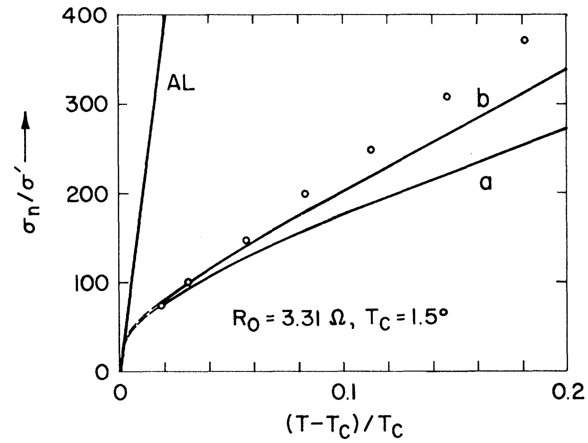

FIG. 2. The inverse of the excess conductivity of an aluminum film. Curve a is the sum of Eqs. (1) and (2) with δ given by Eq. (6). Curve b includes additional corrections. The experimental points have been taken from Ref. 5.

with δ_1 given by Eq. (6). There are no adjustable parameters. The curve is shown as a broken line in the critical region where it is no longer accurate. There is, however, an important correction to be made at higher temperatures. Equation (2) arises from terms in the Maki diagram where the fluctuation frequency ω_s is zero. Terms with $\omega_s \neq 0$ are also infinite when $\delta = 0$, but do not have a strong temperature dependence for constant δ. Since our δ_1 decreases rapidly with increasing temperature, these additional terms become temperature dependent and important. As a rough estimate of the effects of these terms, we have summed them numerically using the uncorrected δ_1 of Eq. (6). The result of this correction is curve b, which again with no adjustable parameters gives improved agreement with experiment. A similarly good fit has also been obtained for a film[6] with $R_\square = 129\ \Omega$, and should be found with all films which can be fitted by Eq. (2) with a temperature-independent value of δ of the order of $5\times 10^{-4} R_\square$. It should also be noted that the presence of a small additional pair-breaking effect will serve to push our curves up and enhance the agreement with experiment.

Finally, the effective pair-breaking strength we compute is far too small to account for the results on Bi, Pb, and Ga films.[2] It may be that electron-phonon interactions serve to enhance the pair-breaking due to fluctuations in these presumably strong-coupling materials.

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Fluctuation Theory of the Superconducting Transition in Restricted Dimensionality*

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The results of a calculation of the properties of a dirty superconductor near the transition temperature are described. The density of states and conductivity are obtained within an approximation that corresponds to BCS theory above T_c. In two dimensions or less no singularity is encountered at T_c; instead, the conventional BCS results are obtained for $T \ll T_c$, even though the order parameter is strictly zero. The anomalous contribution to the conductivity reported by Maki is found to be well behaved and finite in the absence of external pair breaking.

In this Letter we give a simplified derivation of several results of a self-consistent microscopic calculation[1] of the effects of fluctuations on the properties of a superconducting alloy near the transition. The electrons are taken to interact via the usual pairing term $V(k-k')=-|V|$ and with a random distribution of impurities. The proper inclusion of the finite lifetime of the interacting electrons leads to a finite dc conductivity, in contrast to a previous calculation.[2]

Just above the transition, resonance scattering occurs between electrons of opposite spin and momentum, foreshadowing the formation of bound pair states. The anomalously large contribution of such repeated scattering must be included from the outset by replacing the bare interaction V with the t matrix, which within mean-field theory has the form

$$t^{-1}(p,\nu)=N_0[\epsilon+(\pi|\nu|/8T+Dp^2)], \tag{1}$$

where $\epsilon=\ln(T/T_c)$, $D=v_F l/3$ is the diffusion constant, and N_0 is the density of states at the Fermi surface. As an effective electron-electron interaction, the t matrix leads to the following decay rate for the electron states ($\hbar=k_B=1$, d = dimensionality):

$$\frac{1}{\tau_n}=2\pi\sum_{p,\nu}\left|\frac{t(p,\nu)}{2}\right|^2 N_0 \propto \epsilon^{-2+d/2}. \tag{2}$$

The decay rate (2) may be regarded as the rate at which electrons leave a "normal fluid" to combine in a "superfluid" of virtual pair states at an energy above the Fermi surface. The inverse rate at which electrons decay from the superfluid back to the Fermi surface is given by the time-dependent Ginzburg-Landau equation:[3] $1/\tau_s=8T\epsilon/\pi$. A simple detailed-balance argument gives the (small) change in the density of states $N(0)$ at the Fermi surface:[4]

$$\frac{N(0)-N_0}{N_0}=\frac{\tau_s}{\tau_n}\simeq\left(\frac{\epsilon}{\epsilon_d}\right)^{-3+d/2}. \tag{3}$$

We use Eq. (3) to define a characteristic reduced temperature ϵ_d at which the relative change in $N(0)$ becomes of order unity. The temperature