curve in Figs. 2(f) and 3(f) is the net potential jump φ, as a function of M_s.

These equations predict an upper critical Mach number at which $n_r = n_f$ and above which one does not have a shock but a pure piston. In the magnetic case this critical Mach number is 3.18 and in the electrostatic case 6.5. These equations implicitly assume that the peak potential in the shock front is just large enough to reflect ions to provide the necessary dissipation. Thus they have solutions with n_r finite down to $M_s = 1$. Solutions with $n_r = 0$ are not possible except at $M_s = 1$. However from Figs. 2(f) and 3(f) we see that as M_s approaches the lower critical Mach number, nonsteady solitary-wave solutions are possible with peak potentials too small to reflect ions. The simulations, particularly for the magnetic case, illustrate a distinct transition from solitary-wave-like to shocklike behavior right at the lower critical Mach number. For moderately high Mach numbers good agreement is seen with the shock-wave theory where the dissipation due to trapped and reflected ions becomes sufficiently strong to produce steady-state behavior. A finite fraction of ions are reflected in the limit of $T_i \to 0$ because of regular fluctuations in the shock front. For this reason these solutions cannot be obtained by the stationary quasipotential method.[6] In the electrostatic case, if an isothermal electron equation of state is used, the upper critical Mach number drops to 1.82, which perhaps explains why laboratory electrostatic shocks[7] in which the electrons are maintained nearly isothermal do not approach as large a Mach number as obtained in the simulations. Above the upper critical Mach number (and prior to shock formation in the electrostatic case) the simulations show a pure piston whose properties are given by $n_r = n_f$ and Eq. (9). Again the system is closed by using the appropriate equation of state.

The authors would like to thank Mr. G. Beeler, Jr., for performing the electrostatic computer simulations. This work was performed under the auspices of the U. S. Atomic Energy Commission.

[1]D. W. Forslund and C. R. Shonk, Phys. Rev. Lett. 25, 1699 (1970).

[2]R. Z. Sagdeev, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1966), Vol. 4, p. 23.

[3]P. L. Auer, H. Hurwitz, and R. W. Kilb, Phys. Fluids 4, 1105 (1961), and 5, 298 (1962).

[4]R. L. Morse and C. W. Nielson, Phys. Fluids 12, 2418 (1969).

[5]C. S. Gardner and G. K. Morikawa, Courant Institute of Mathematical Sciences Report No. NYO-9082, 1960 (unpublished); H. Washimi and T. Taniuti, Phys. Rev. Lett. 17, 996 (1966).

[6]S. S. Moiseev and R. Z. Sagdeev, J. Nucl. Energy: Part C 5, 43 (1963).

[7]R. J. Taylor, D. R. Baker, and H. Ikezi, Phys. Rev. Lett. 24, 206 (1970).

Exact Solution of the Korteweg–de Vries Equation for Multiple Collisions of Solitons

Ryogo Hirota
RCA Research Laboratories, Tokyo International 100-31, Japan
(Received 17 September 1971)

An exact solution has been obtained for the Korteweg–de Vries equation for the case of multiple collisions of N solitons with different amplitudes.

An exact solution has been obtained for the Korteweg–de Vries equation,

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1}$$

with the associated boundary condition $u(x, t) = 0$ at $x = \pm\infty$. The solution is valid for the case of multiple collisions of N solitons (N being an arbitrary but finite integer) with different amplitudes and can be written

$$u(x, t) = -2\frac{\partial^2}{\partial x^2}\ln f(x, t), \tag{2}$$

$$f(x, t) = \det|M|, \tag{3}$$

where the $N \times N$ matrix M has the form

$$M_{ij}(x,t) = \delta_{ij} + \frac{2(P_i P_j)^{1/2}}{P_i + P_j} \exp[\tfrac{1}{2}(\xi_i + \xi_j)], \tag{4}$$

$$\xi_i = P_i x - \Omega_i t - \xi_i^{\,0}, \tag{5}$$

$$\Omega_i = P_i^{\,3}, \tag{6}$$

and where P_i and $\xi_i^{\,0}$ are arbitrary constants which determine the amplitude and phase, respectively, of the ith soliton. The P_i are assumed to be all different. It should be noted that the functional form of u in Eq. (2) is the same as that of Kay and Moses's expression for the reflectionless potential for the one-dimensional Schrödinger equation.[1,2]

Substituting the solution into the original equation to demonstrate its validity gives the following equation for $f(x,t)$:

$$ff_{xt} - f_t f_x + f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^{\,2} = 0, \tag{7}$$

where subscript notation has been used to indicate the partial differentiation. We rewrite f in the following form[3] to prove Eq. (7):

$$f = 1 + \sum_{n=1}^{N} \sum_{{}_N C_n} a(i_1, i_2, \cdots, i_n) \exp(\xi_{i_1} + \xi_{i_2} + \cdots + \xi_{i_n}), \tag{8}$$

$$a(i_1, i_2, \cdots, i_n) \prod_{k<l}^{(n)} a(i_k, i_l), \tag{9}$$

$$a(i_k, i_l) = (P_{i_k} - P_{i_l})^2 / (P_{i_k} + P_{i_l})^2, \tag{10}$$

where ${}_N C_n$ indicates summation over all possible combinations of n elements taken from N, and (n) indicates the product of all possible combinations of the n elements (with the specified condition $k<l$, as indicated).

Substitution of this expression for f into Eq. (7) shows that f is a solution of Eq. (7) provided that the following relation holds:

$$\sum_{l=0}^{n} \sum_{{}_n C_l} a(i_1, i_2, \cdots, i_l) a(i_{l+1}, \cdots, i_n) g(-i_1, -i_2, \cdots -i_l, i_{l+1}, \cdots, i_n) = 0, \tag{11}$$
$$n = 1, 2, \cdots, N,$$

where

$$\begin{aligned} g(-i_1, -i_2, \cdots, -i_l, i_{l+1}, \cdots, i_n) = {} & (-P_{i_1} - P_{i_2} - \cdots - P_{i_l} + P_{i_{(l+1)}} + \cdots + P_{i_n}) \\ & \times [(-P_{i_1} - P_{i_2} - \cdots - P_{i_l} + P_{i_{(l+1)}} + \cdots + P_{i_n})^3 \\ & - (-P_{i_1}^{\,3} - P_{i_2}^{\,3} - \cdots - P_{i_l}^{\,3} + P_{i_{(l+1)}}^{\,3} + \cdots + P_{i_n}^{\,3})], \end{aligned} \tag{12}$$

and $a(i_1) \equiv a(i_0) \equiv 1$.

For a given value of n, Eq. (11) can be transformed into the following identity:

$$\sum_{\sigma_1, \sigma_2, \cdots, \sigma_n = \pm 1} b(\sigma_1 P_1, \sigma_2 P_2, \cdots, \sigma_n P_n) g(\sigma_1 P_1, \sigma_2 P_2, \cdots, \sigma_n P_n) = 0, \tag{13}$$

where

$$b(\sigma_1 P_1, \sigma_2 P_2, \cdots, \sigma_n P_n) = \prod_{k<l}^{(n)} (\sigma_k P_k - \sigma_l P_l)^2, \tag{14}$$

the summation being over all possible combinations of $\sigma_1 = \pm 1$, $\sigma_2 = \pm 1$, $\cdots$, $\sigma_n = \pm 1$.

The identity can be proved by mathematical induction. Let the left-hand side of Eq. (13) be represented by $D(P_1, P_2, \cdots, P_n)$, which is found to have the following properties: (i) D is a symmetric, homogeneous polynomial; (ii) D is an even function of $P_1, P_2, \cdots, P_n$; (iii) if $P_k = P_l$, then

$$D(P_1, P_2, \cdots, P_n) = 2(2P_k)^2 D(P_1, P_2, \cdots, P_{k-1}, P_{k+1}, \cdots, P_{l-1}, P_{l+1}, \cdots, P_n) \prod_{m=1}^{n}{}' (P_k^{\,2} - P_m^{\,2})^2,$$

where the prime indicates that the product is taken over $m = 1$ to n, except for $m = k$ and $m = l$.

The identity is easily verified for $n = 1$ and 2. Now, assume that the identity holds for $n - 2$. Then, relying on properties (i), (ii), and (iii), it is seen that D can be factored by a symmetric homogeneous polynomial

$$\prod_{k<l}^{(n)} (P_k^2 - P_l^2)^2$$

of degree $2n(n-1)$. On the other hand, Eq. (13) shows the degree of D to be $n(n-1)+4$. Hence, D must be zero for n, and the identity holds.

[1]I. Kay and H. E. Moses, J. Appl. Phys. 27, 1503 (1956).
[2]C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. 19, 1095 (1967).
[3]J. Rubinstein, J. Math. Phys. 11, 258 (1970).

Energy Loss of H, D, and ^{4}He Ions Channeled Through Thin Single Crystals of Silicon*

G. Della Mea, A. V. Drigo, S. Lo Russo, and P. Mazzoldi
Istituto di Fisica dell'Università, Unità del Gruppo Nazionale di Struttura della Materia, Padova, Italy

and

G. G. Bentini
Laboratorio per la Chimica e Tecnologia dei Materiali e Componenti per l'Elettronica, Consiglio Nazionale dell Ricerche, Bologna, Italy

(Received 28 July 1971)

The energy loss of ions channeled through the ⟨111⟩ Si channel is studied in the energy range 0.9 to 5.0 MeV. The energy dependence of the ratio between channeling and random stopping power above 3 MeV shows an increase which can be interpreted in terms of core-electron excitation. The velocity dependence of the channeling stopping power is also studied.

In recent years many experiments have been performed on the energy loss of light ions channeled through semiconductor single crystals in the energy range above 3 MeV.[1-3] The results have been discussed in terms of localized and nonlocalized contributions to the electronic stopping power.[2] Appleton, Erginsoy, and Gibson[1] used their results to extract the local density of valence electrons, sampled by the well-channeled protons along the Si ⟨110⟩ axial direction. The value they obtained in this way was about 4.

In the present work we extend the energy-loss measurements to the lower energy region, using H^+, D^+, and $^4He^+$ ions. Our aim is to check (a) the channeling energy loss in the energy range where the incident ions interact only with the weakly bounded valence electrons; (b) the energy threshold for the core-electron contribution to the stopping power; (c) the mass dependence of the channeling energy loss at low energy; and (d) the energy dependence of the ratio between channeling and random stopping power. In this Letter we present some preliminary results concerning the Si ⟨111⟩ axial direction.

The incident beam, obtained from the 5.5-MeV Van de Graaff accelerator of Laboratori Nazionali Legnaro, was collimated by annular collimators of various sizes and arranged at given distances. The minimum hole was 0.3 mm in diameter and the maximum divergence of the beam was always kept better than 0.1°. The thickness of the targets ranged from 1.5 to 32 μm. The thickness of the samples was carefully checked by the energy loss of the transmitted particles in random conditions, using the tabulated stopping power.[4]

In the case of ^{4}He ions, corrections to these values, due to a reduction in the effective charge of the particles, could also be considered, as suggested by Bloom and Sauter.[5] Since, however, only a few experimental data are available at present on this point, we preferred to follow Williamson's treatment which, in addition, yielded a measured value for the thickness of the