[17]The transverse-momentum distributions are quite similar for small $p_{\parallel}$ in the different data samples.

[18]For comparison, corresponding values in Reaction (3) are 0.396±0.013, 0.302±0.009, and 0.319±0.015, respectively, at 8, 18.5, and 24.8 GeV/c. The errors here and in the figures include only statistical errors on the distributions and on the cross-section determinations. Some allowance should be made for systematic uncertainties of approximately a few percent in the evaluations of cross sections, where somewhat different procedures (see Ref. 14) had to be used with the different samples.

[19]The shapes differ at larger $p_{\parallel}$, the 18.5-GeV/c distribution falling less rapidly with increasing $p_{\parallel}$ than the 7-GeV/c distribution.

[20]The fit to the data for $p_{\parallel}<0$, where target fragmentation is most surely expected to dominate, has a χ^2 probability of 8.2% with no allowance for systematic effects such as differences in normalization of cross sections.

[21]As $p_{\parallel}$ is varied, systematic effects can be observed in Figs. 3 and 4 which indicate the limitations of the Regge model in its present form. For higher incident momenta the particle densities decrease less rapidly as $p_{\parallel}$ increases. This is consistent with the presence of significant contributions at $p_{\parallel} \gtrsim 0.6$ GeV/c from sources other than target fragmentation which become larger at higher energies.

Two-Component Poincaré-Invariant Equations for Massive Charged Leptons*

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(Received 28 July 1971)

An alternative to Dirac's factorization of the Klein-Gordon equation is developed; the resulting two-component, $m\neq 0$ equations are proved Poincaré invariant. The particles interact *chirally* and *minimally* with the electromagnetic field ($g=2$). Our equations yield factorizations of Kramers's equations and a conserved chiral current (without γ_5 projection) to implement in a natural way the Feynman–Gell-Mann approach to weak interactions. Besides possessing sharp chirality, the particles possess a new dichotomic quantum number.

The theory of spin-$\frac{1}{2}$ particles was constructed by Dirac by means of a factorization of the Klein-Gordon equation over the field of four-component spinors. Wigner's subsequent analysis of the irreducible representations (irreps) of the Poincaré group provided a secure general foundation and extension of Dirac's construction. It is a familiar result of this analysis that a two-component spin-$\frac{1}{2}$ irrep (particle) can be characterized only by sharp CP and not P or C separately. The neutrinos ($m=0$) provide a well-known physical example.

It is commonly believed that for mass $m\neq 0$ Dirac's construction is unique, and that, in particular, a two-component particle having $m\neq 0$ cannot possess a first-order Poincaré-invariant wave equation. We will show that this belief is incorrect; we shall explicitly construct Poincaré-invariant first-order wave equations for two-component massive charged particles, characterized by chirality and a second new quantum number.

Let us note that our construction in no way contradicts Dirac's work. Dirac explicitly states that the uniqueness of his construction hinged on a fundamental assumption: that the matrices entering the factorization are to be *independent of space-time*, that is, *they are to represent independent new degrees of freedom*. This assumption is omitted in our construction.

That the factorizing matrices now involve space-time complicates the proof of invariance; accordingly we first applied these equations to an external Coulomb field[2] which—by singling out a particular point and a particular Lorentz frame—makes invariance questions irrelevant. We found precisely the usual Dirac-Coulomb levels, with this distinction: The new quantum number splits the (degenerate) spectrum into two *nondegenerate* spectra (omitting spatial degeneracy).

The Feynman–Gell-Mann theory of the Fermi interaction[3] began from the iterated Dirac equation in the presence of arbitrary electromagnetic fields, that is,

$$(\gamma\cdot\Pi+m)(\gamma\cdot\Pi-m)\psi =[\Pi\cdot\Pi+e\vec{\sigma}\cdot(i\rho_1\vec{E}+\vec{B})-m^2]\psi=0, \quad (1)$$

where

$$\gamma\equiv(\rho_3,\rho_2\vec{\sigma}),\quad \Pi\equiv(p_0-eA_0,\vec{p}-e\vec{A}).$$

We may project onto states of sharp chirality splitting Eq. (1) into the two (two-component) equations first given by Kramers and by van der Waerden.[4] These equations are, for $\rho_1\to+1$,

$$(\Pi^-\Pi^+-m^2)\varphi=0; \quad (2a)$$

for $\rho_1\to-1$,

$$(\Pi^+\Pi^--m^2)\varphi=0; \quad (2b)$$

here

$$\Pi^\pm\equiv\Pi_0\pm\vec{\sigma}\cdot\vec{\Pi}. \quad (2c)$$

Note that Eqs. (2a), 2(b) are distinguished by chirality and are distinct only in the presence of electromagnetic fields. The Klein-Gordon (KG) equation is the field-free special case.

Let us introduce in the KG equation the two-component space-time-dependent anticommuting operators[5] defined by

$$\eta_1\equiv\vec{\sigma}\cdot\hat{r},\quad \eta_2\equiv i\eta_1\eta_3,\quad \eta_3\equiv(-)^{j(\mathcal{K})+1/2}S(\mathcal{K}), \quad (3)$$

where

$$\mathcal{K}\equiv-(\vec{\sigma}\cdot\vec{L}+1),\quad S(\mathcal{K})\equiv\frac{\mathcal{K}}{|\mathcal{K}|}=-\frac{\vec{\sigma}\cdot\vec{L}+1}{j(\mathcal{K})+\frac{1}{2}}. \quad (4)$$

The operator $\mathcal{K}$ is a two-component form of Dirac's K quantum number; since the eigenvalues, $\mathcal{K}\to\kappa$, are the (±) integers *excluding* zero, (4) is well defined. The operator $j(\mathcal{K})$ has eigenvalues $j=|\kappa|-\frac{1}{2}$, and satisfies $\vec{J}^2\to j(j+1)$.

The significance of the η_i is that they obey $\eta_i\eta_j=ie_{ijk}\eta_k$. The operator η_3 *has the special significance that it is an explicit space-time-dependent form of the parity operator acting in the particle configuration space.* This allows a factorization of the KG equation,

$$(p_0^2-\vec{p}^2-m^2)=(p_0-\vec{\sigma}\cdot\vec{p}-\eta_3m)\times(p_0+\vec{\sigma}\cdot\vec{p}+\eta_3m). \quad (5a)$$

To see this note that $p_0\,[\equiv i(\partial/\partial t)]$ commutes with $\vec{\sigma}\cdot\vec{p}$, and with η_3, since (1) these operators have no *explicit* time dependence, and (2) p_0 is a *true scalar* operator (even parity). That the operators $\vec{\sigma}\cdot\vec{p}$ and η_3 anticommute can be shown directly,[5] but results more simply from noting that $\vec{\sigma}\cdot\vec{p}$ is *pseudoscalar*. A more convenient form of (5a) results if we multiply on both sides by η_3:

$$p_0^2-\vec{p}^2-m^2=[\eta_3(p_0-\vec{\sigma}\cdot\vec{p})+m]\times[\eta_3(p_0-\vec{\sigma}\cdot\vec{p})-m]. \quad (5b)$$

To incorporate general electromagnetic fields we use gauge invariance:

$$p_0\pm\vec{\sigma}\cdot\vec{p}\to\Pi_0\pm\vec{\sigma}\cdot\vec{\Pi}=\Pi^\pm. \quad (6)$$

The factorization given in (5b) implies the wave equations $(\rho_1\to-1)$

$$[\eta_3(\Pi_0+\rho_1\vec{\sigma}\cdot\vec{\Pi})\pm m]\varphi=0. \quad (7a)$$

The case $\rho_1\to+1$ is also allowed, hence (7a) is true in general. For $A_\mu=0$, we have

$$[\eta_3(p_0+\rho_1\vec{\sigma}\cdot\vec{p})\pm m]\varphi=0. \quad (7b)$$

To prove that (7a) implies a factorization of (2a), (2b), consider the product $(\rho_1\to1)\;[\eta_3(\Pi_0+\vec{\sigma}\cdot\vec{\Pi})+m][\eta_3(\Pi+\vec{\sigma}\cdot\vec{\Pi})-m]$. First note that η_3 commutes with eA_0 since A_0, like p_0, is a true scalar, hence $[\eta_3,\Pi_0]=0$. The crucial step is to show that $[\eta_3,\vec{\sigma}\cdot\vec{\Pi}]_+=0$, or equivalently, $[\eta_3,\vec{\sigma}\cdot\vec{A}]_+=0$. To verify this consider first a constant magnetic field: $\vec{A}=\frac{1}{2}(\vec{B}\times\vec{r})$, where $\vec{B}$ is a constant axial vector. The odd-parity information in $\vec{A}$ is now *explicitly* carried by $\vec{r}$ and η_3 as a particle-space parity operator "sees" the odd parity: $[\eta_3,\vec{\sigma}\cdot\frac{1}{2}(\vec{B}\times\vec{r})]_+=0$. By superposing sufficiently many locally constant fields, and using algebraic linearity, we obtain the general case, i.e., $[\eta_3,\vec{\sigma}\cdot\vec{A}]_+=0$ for a general polar vector $\vec{A}$. To incorporate the general situation more expediently, we extend the meaning of η_3 to include *implicit* parity information. Thus, η_3 has henceforth the significance of a *general* parity operator.[6]

We have thus factorized Kramers equation for general electromagnetic fields; reintroducing ρ_1 shows that we have, in fact, a factorization of Eq. (1) itself—a factorization which splits into chiral two-component wave equations.

The four cases for Eq. (7) *each possess a conserved current:*

$$j\equiv(\varphi^\dagger[\Pi,\rho_1\vec{\sigma}]\varphi),\quad \partial\cdot j=0. \quad (8)$$

The proof of Poincaré invariance for (7) is rather delicate. The difficulty stems from two sources: the space-time dependence of the η's and the discreteness of reflection operators.

For Lorentz invariance we can restrict attention to proving the invariance of $\eta_3(p_0+\vec{\sigma}\cdot\vec{p})$ for Lorentz boosts. The boosts are generated by the polar vector operator $\vec{K}\equiv i(\vec{N}+\frac{1}{2}\vec{\sigma})$, where $\vec{N}\equiv M_{k4}$, $M_{\mu\nu}\equiv e_{\mu\nu\alpha\beta}x_\alpha p_\beta$. A *finite* boost is generated by $V \equiv (i\vec{v}\cdot\vec{K})$. Note that $\vec{v}$ is a *constant polar vector* parametrizing the boost velocity. Hence,

$$\eta_3(p_0+\vec{\sigma}\cdot\vec{p}) \equiv \eta_3 P_\mu \sigma_\mu \to V^{-1}(\eta_3 p_\mu \sigma_\mu)V. \qquad (9)$$

Since $[\vec{\sigma},\vec{N}]=0$, we have $V=UW=WU$, where $U=\exp(-\vec{v}\cdot\vec{N})$, $W=\exp(-\frac{1}{2}\vec{v}\cdot\vec{\sigma})$. Observe now that $\vec{v}\cdot\vec{N}$ has *even* parity; *hence U commutes with* η_3 and generates on p the finite transformation $p_\mu \to a_{\mu\nu}p_\nu$. By contrast, the *odd-parity* operator $\vec{v}\cdot\vec{\sigma}$ *anticommutes* with η_3; hence $W^{-1}\eta_3=\eta_3 W$. Precisely as required, then, the σ_μ transform as $W\sigma_\mu W = a_{\mu\tau}\sigma_\tau$. By definition, $a_{\mu\nu}a_{\mu\tau}=\delta_{\nu\tau}$; Lorentz invariance follows.

Accordingly we have shown that in (7) the *chiral* four-vectors, $\eta_3(\Pi,\rho_1\vec{\sigma})$, enter in the scalar product with p_μ; moreover by comparison with the current, (8), we see that the analog to the Pauli adjoint $\bar{\psi}$ is here $\bar{\varphi}\equiv\varphi^\dagger\eta_3$. (Using this inner product one can verify invariance in an alternative way.)

Consider the translational invariance of (7). Again we consider finite transformations, the finite space-time displacement operator being $D\equiv\exp(i\vec{d}\cdot\vec{p})$. We need consider only the behavior of η_3. Note now that $\vec{d}\cdot\vec{p}$ has in every case *even parity*; clearly η_3 commutes and the equations are Poincaré invariant.

{To appreciate the delicate nature of these arguments note that the infinitesimal generator of spatial displacements, $\vec{p}$, does *not* commute with (7b) and hence is *not* constant under the Hamiltonian. Yet the equations *are* displacement invariant! This "paradox" was avoided by considering finite transformations; more generally we see the necessity of defining infinitesimal boosts, say, by

$$\mathcal{O}\to\mathcal{O}'\equiv\mathcal{O}+\delta\mathcal{O}, \qquad \delta\mathcal{O}\equiv[\delta\vec{v}\cdot\vec{K},\mathcal{O}], \qquad (10)$$

and the latter commutator is *not* necessarily the same as $\delta\vec{v}\cdot[\vec{K},\mathcal{O}]$, despite the fact that $\delta\vec{v}$ is a parametric constant vector. The peculiar phenomena inherent to (7) are associated with a new phenomenon: rapid fluctuations $t\approx m^{-1}$ between positive- and negative-parity states.}

Although Eq. (7b) does indeed possess a rest frame, one must be careful to interpret this concept properly, since *the rest frame is qualitatively distinct from a general frame in the following ways:* (1) In every frame, except the rest frame, $\vec{p}$ fluctuates and parity is *not* sharp; (2) half of the solutions to (7b) *vanish* in the rest frame; (3) rest-frame solutions properly belong the the space of KG solutions. To avoid any possible inconsistency one may imagine $\vec{p}$ to contain an infinitesimal imaginary part $i\epsilon$, which then regularizes the system. [More physically one may imagine (for continuum states) an infinitesimally weak Coulomb field—which accomplishes the same end.]

Following convention, we must now designate the new quantum number contained in (7a). To rewrite (7a) we first define a dichotomic operator[7]:

$$\varsigma\equiv m^{-1}\eta_3(\Pi_0+\rho_1\vec{\sigma}\cdot\vec{\Pi}), \quad (\varsigma)^2=\Pi. \qquad (11)$$

Equation (7a) assumes the form $\varsigma\varphi\to\pm\varphi$. (If we wish to include the neutrino we can extend this operator to three values by defining stigma $\to 0$ in this case.)

Concluding remarks.—(a) It has been shown[4] that Kramers equation has the remarkable property of allowing only $g=2$. This property clearly accords with (7a), where one observes that there simply are no operators—analogous to $[\gamma_\mu,\gamma_\nu]$ —available to form an invariant from $F_{\mu\nu}$. *Thus, this system is inherently characterized by minimal electrodynamics* and correspondingly the g factor is *exactly two*. It seems reasonable in view of this fact to designate (7) as a sort of "primitive" lepton equation, describing four types of $m\neq 0$ leptons ($\rho_1\to\pm 1$, $\varsigma\to\pm 1$) and two types of $m=0$ leptons ($\rho_1\to\pm 1$). (b) It will be observed that we have *deduced the existence of "leptonic" conserved chiral currents*, Eq. (8), which, following Feynman and Gell-Mann, should be the appropriate basis for a current-current theory of the Fermi interaction. (c) The *physical* massive leptons have $(g-2)/2\approx\alpha/2\pi$. Assuming that the electron and muon are almost pure "leptons," we conclude that any chirality breaking in purely leptonic decays is of order $\alpha/2\pi$, i.e., $|V/A|-1\approx\alpha/2\pi$ in muon decay.

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*Work supported in part by the National Science Foundation and the U. S. Army Research Office, Durham.

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[2]L. C. Biedenharn and M. Y. Han, to be published.

[3]R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958).

[4]L. M. Brown, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Interscience, New York, 1962), Vol. IV.

[5]A detailed discussion (and original references) may

be found in L. C. Biedenharn and P. J. Brussaard, *Coulomb Excitation* (Oxford U. Press, Oxford, England, 1965).

[6]This step was in error in Ref. 2; we wrongly argued that $\vec{\sigma}\cdot\vec{A}$ was a *pseudoscalar* under $\vec{J}=\vec{L}+\vec{\sigma}/2$, hence "$\eta_3$" $=S(\mathcal{K})$ anticommutes. In fact, $\vec{\sigma}\cdot\vec{A}$ is a *polar vector* under $\vec{J}$ and to achieve anticommutation we must redefine η_3 as given in (3). The use of the incorrect operator for η_3 is responsible for our statement in Ref. 2 that Lorentz invariance fails.

[7]The Greek letter ς is called "stigma" [see O. Neugebauer, *The Exact Sciences in Antiquity* (Princeton U. Press, Princeton, N. J., 1952), pp. 10 and 24], with the quantum number being a distinguishing "mark."